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ON THE BOUNDED CLOSURE OF THE RANGE OF AN OPERATOR

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Abstract. The "bounded closure of the range" of an operator between two normed spaces is a linear subspace lying between the range and its closure. The induced concept of "almost onto" is a sort of first draft of the concept of "almost open".

If $T \in BL(X,Y)$ is bounded and linear between normed spaces then the range of T is a linear subspace of Y:

$$T(X) = \{Tx : x \in X\}. \tag{0.1}$$

Since $T(X)$ may or may not be closed we must also consider its closure:

$$\mathrm{cl}\ T(X) = \{\lim_n Tx_n : x \in X^{\mathbf{N}} \text{ with } Tx \in c_1(Y)\}. \tag{0.2}$$

Here we write $c_1(Y)$ for the convergent sequences in Y, and $c_0(Y)$ for the null sequences:

$$c_0(X) \subseteq c_1(X) \subseteq \ell_\infty(X) = \{x \in X^{\mathbf{N}} : \sup_n ||x_n|| < \infty\}. \tag{0.3}$$

1. Definition. If $T \in BL(X,Y)$ is bounded and linear between normed spaces then the bounded closure, or almost closure, of its range is the set

$$\mathrm{cl}^\sim(T,X) = \{\lim_n Tx_n : x \in \ell_\infty(X) \text{ with } Tx \in c_1(Y)\}. \tag{1.1}$$

We shall describe T as having bounded closure, or being relatively almost onto, if there is equality

$$\mathrm{cl}^\sim(T,X) = \mathrm{cl}\ T(X), \tag{1.2}$$

and as being boundedly closed, or almost closed, if there is equality

$$T(X) = \mathrm{cl}^\sim(T,X). \tag{1.3}$$

Evidently the bounded closure is a linear subspace, with

$$T(X) \subseteq \mathrm{cl}^\sim(T,X) \subseteq \mathrm{cl}\ T(X). \tag{1.4}$$

Like the range and its closure, the bounded closure is unchanged when the operator is made to be one-one:

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2. Theorem. *If* $T \in BL(X,Y)$ *there is equality*

$$\mathrm{cl}^{\sim}(T^{\wedge}, X/T^{-1}(0)) = \mathrm{cl}^{\sim}(T,X). \tag{2.1}$$

Proof. Inclusion one way is obvious: if $Tx_n \to y$ with $\sup_n ||x_n|| < \infty$ then certainly

$$Tx_n \to y \text{ with } \sup_n \mathrm{dist}(x_n, T^{-1}(0)) < \infty; \tag{2.2}$$

conversely if (2.2) holds we may find (x'_n) with

$$x'_n - x_n \in T^{-1}(0) \text{ and } ||x'_n|| \le \mathrm{dist}(x_n, T^{-1}(0)) + 1 : \tag{2.3}$$

but now $y = \lim_n Tx'_n$ with $x' \in \ell_\infty(X)$. □

We shall describe $T \in BL(X,Y)$ as *almost onto* if there is equality

$$\mathrm{cl}^{\sim}(T,X) = Y : \tag{2.4}$$

3. Theorem. *If* $T \in BL(X,Y)$ *there is implication*

$$T \textit{ onto} \Longrightarrow T \textit{ almost onto} \Longrightarrow T \textit{ dense}, \tag{3.1}$$

$$T \textit{ almost open} \Longrightarrow T \textit{ almost onto} \Longrightarrow T \textit{ dense}, \tag{3.2}$$

and hence also

$$T \textit{ relatively almost open } \Longrightarrow T \textit{ has bounded closure.} \tag{3.3}$$

Proof. The common second implication is clear. If $T \in BL(X,Y)$ is onto then

$$y \in Y \Longrightarrow y = Tx = \lim_n Tx_n \text{ with } x_n = x \ (n \in \mathbf{N}), \tag{3.4}$$

finishing the proof of (3.1). If $T \in BL(X,Y)$ is almost open then there is $k > 0$ for which

$$y \in Y \Longrightarrow y \in \mathrm{cl}\{Tx : ||x|| \le k||y||\}, \tag{3.5}$$

giving the condition (1.1) . This proves (3.2); for (3.3) we apply (3.2) with Y replaced by cl $T(X)$, noting that if $T^{\vee} : X \to T(X)$ is almost open then so is the related operator $T^{\vee\vee} : X \to \mathrm{cl}\ T(X)$. To make this last observation note that if $Z \subseteq Y$ is a linear subspace and $y \in \mathrm{cl}\ Z$ then ([5]; [6] Theorem 1.5.1) there is (z_n) in Z for which

$$z_n \to y \text{ with } ||z_n|| \le ||y||. \tag{3.6}$$

□

For example if T is bounded below then it has bounded closure; in particular the "bounded closure" of a subspace $X \subseteq Y$ is just the closure.

It is familiar that the first implications of neither (3.1) nor (3.2) can be reversed: if for example $T : X \to Y$ is the embedding of a dense proper subspace then ([5] Theorem 2.1; [6] Theorem 4.7.2) T is almost open but not onto, while if $T : X \to Y$ presides over the change of norm from a complete space X to the same space with a strictly weaker norm then ([5] Theorem 2.2; [6] Theorem 4.7.3) T is onto but not almost open. For an example in which T is dense but not almost onto take $X = Y = c_0$ and $T = W : (x_n) \mapsto (x_n/n)$: then if $y_n = 1/\sqrt{n}$ we find

$$||y - Tx_n|| \le \frac{1}{n} \Longrightarrow \sqrt{n} - 1 \le |x_{nn}| \le \sqrt{n} + 1, \tag{3.7}$$

so that $||x_n|| \ge \sqrt{n} - 1$ and $x = (x_n) \notin \ell_\infty(c_0)$. More generally, the first implication in (3.1) cannot be reversed, even when the space X is complete:

4. Example. If $e = (e_n) \in c_0$ and $T : X \to Y$ is given by setting

$$X = Y = c_0 \text{ and } (Tx)_n = e_n x_n \ (n \in \mathbf{N},\ x \in c_0), \tag{4.1}$$

then

$$T(X) = \{(e_n u_n) : u = (u_n) \in c_0\}, \tag{4.2}$$

$$\mathrm{cl}^{\sim}(T, X) = \{(e_n u_n) : u = (u_n) \in \ell_\infty\} \tag{4.3}$$

and

$$\mathrm{cl}(TX) = \{y \in c_0 : \forall n \in \mathbf{N}, e_n = 0 \Longrightarrow y_n = 0\}. \tag{4.4}$$

Proof. Equality (4.2) is clear, and trivial. For inclusion one way in (4.3) suppose $y_n = e_n u_n$ with $u = (u_n) \in \ell_\infty$ and define $x = (x_m) = (x_{mn})$ by setting

$$x_{mn} = u_n - \tfrac{1}{m} \text{ if } |u_n| \geq \tfrac{1}{m}, \ = 0 \text{ if } |u_n| < \tfrac{1}{m} : \tag{4.5}$$

then

$$|u_n| \geq \tfrac{1}{m} \Longrightarrow |y_n - e_n x_{mn}| = |e_n(u_n - x_{mn})| = |e_n|\tfrac{1}{m} \leq \tfrac{1}{m}||e||_\infty, \tag{4.6}$$

while

$$|u_n| < \tfrac{1}{m} \Longrightarrow |y_n - e_n x_{mn}| = |e_n u_n| \leq \tfrac{1}{m}||e||_\infty; \tag{4.7}$$

in both cases

$$|x_{mn}| \leq |u_n| \leq ||u||_\infty. \tag{4.8}$$

Conversely if $x = (x_m) = (x_{mn})$ exists for which

$$\sup_n |y_n - e_n x_{mn}| \leq \tfrac{1}{m} \text{ with } \sup_{m,n} |x_{mn}| = k < \infty \tag{4.9}$$

then $|y_n| \leq k|e_n| + \frac{1}{m}$ for all m, n and hence $|y_n| \leq k|e_n|$ for all n, giving

$$y_n = e_n u_n \text{ for all } n \text{ with } ||u||_\infty \leq k. \tag{4.10}$$

Inclusion one way in (4.4) follows from the continuity of the functionals $x \mapsto x_n$ on c_0; conversely if $e_n = 0 \Longrightarrow y_n = 0$ then $y - P_m y \to 0$ with $P_m y = (y_1, y_2, \dots, y_m, 0, \dots) \in T(X)$. □

As soon as we have found an operator for which $T(X) \neq \mathrm{cl}^{\sim}(T, X)$ we have an operator which is almost onto but not onto:

5. Theorem. *If* $T = W^{\sim} : c_0 \to \mathrm{cl}^{\sim}(W, c_0)$ *is induced by the weight operator* $W\colon (x_n) \mapsto (\frac{1}{n} x_n)$ *on* c_0 *then* T *is almost onto but not onto.*

Proof. By construction $T = W^{\sim}$ is almost onto; it fails to be onto because the sequence $(\frac{1}{n})$ is in $Y = \mathrm{cl}^{\sim}(W, c_0)$ and not in the range $W(c_0)$. □

The first implication of (3.2) does reverse when the space Y is complete. It is tempting to try a Baire theorem argument with sets of the form

$$R_k^{\sim}(T) = \{y \in Y : y \in \mathrm{cl}\{Tx : ||x|| \leq k||y||\}\} : \tag{5.1}$$

this appears to founder on the lack of additivity among the $R_k^{\sim}(T)$. However the uniform boundedness principle and the Hahn-Banach theorem are available:

6. Theorem. *If* $T \in BL(X, Y)$ *there is implication*

$$Y \textit{ complete, } T \textit{ almost onto} \Longrightarrow T \textit{ almost open.} \tag{6.1}$$

Proof. If $T : X \to Y$ is almost onto then

$$\{g(y) : ||gT|| \le 1\} \text{ is bounded for all } y \in Y : \tag{6.2}$$

for if $y \in Y$ there is $x \in \ell_\infty(X)$ with $Tx_n \to y$, giving

$$|g(y)| = \lim_n |gTx_n| \le ||gT||||x||_\infty. \tag{6.3}$$

By uniform boundedness ([6] Theorem 4.9.1), using the completeness of Y, there is $k > 0$ for which

$$||gT|| \le 1 \Longrightarrow ||g|| \le k, \tag{6.4}$$

so that $||g|| \le k||gT||$. This makes the dual operator $T^\dagger : Y^\dagger \to X^\dagger$ bounded below, and hence by Hahn-Banach separation ([6] Theorem 5.5.2) $T : X \to Y$ almost open. □

The bounded closure of the range can be obtained from the range of the "enlargement":

7. Theorem. *If* $T \in BL(X, Y)$ *then*

$$\mathrm{cl}^\sim(T, X) = \{y \in Y : \mathbf{q}(y) \in \mathbf{Q}(T)\mathbf{Q}(X)\}. \tag{7.1}$$

Proof. Here ([5]; [6] Definition 1.9.2)

$$\mathbf{Q}(X) = \ell_\infty(X)/c_0(X), \tag{7.2}$$

with $\mathbf{q} : X \to \mathbf{Q}(X)$ the natural embedding and $\mathbf{Q}(T)$ the operator induced naturally by T; then, abusing notation,

$$y \in \mathrm{cl}^\sim(T, X) \Longleftrightarrow y \in T\ell_\infty(X) + c_0(Y) \Longleftrightarrow \mathbf{q}(y) \in \mathbf{Q}(T)\mathbf{Q}(X). \quad \square \tag{7.3}$$

From Theorem 7 it follows that if $M \subseteq Y$ is a linear subspace then

$$M \subseteq \mathrm{cl}^\sim(T, X) \Longleftrightarrow M \subseteq T\ell_\infty(X) + c_0(Y) : \tag{7.4}$$

this contrasts with the result of Harte and Shannon [8] that, if the spaces are complete,

$$\mathrm{cl}\ M \subseteq T(X) \Longleftrightarrow \ell_\infty(M) \subseteq T\ell_\infty(X); \tag{7.5}$$

in the special case $M = TX$ Albrecht and Mehta [1] find

$$\mathrm{cl}\ M \subseteq T(X) \Longleftrightarrow \ell_\infty(M) \subseteq T\ell_\infty(X) + c_0(Y). \tag{7.6}$$

The Albrecht/Mehta argument ([6] Theorem 5.7.1) solves Problem (4.1.3) of [5], possibly more directly than the argument of Harte and Mathieu [7]:

8. Theorem. *If* $T \in BL(X, Y)$ *there is implication*

$$\ell_\infty(Y) \subseteq T\ell_\infty(X) + c_0(Y) \iff T \textit{ almost open.} \tag{8.1}$$

Proof. We claim, in the notation of (7.2), that

$$\mathbf{Q}(T) \text{ dense } \Longrightarrow T \text{ almost open} \Longrightarrow \mathbf{Q}(T) \text{ open}. \tag{8.2}$$

The second implication is straightforward ([5] Theorem 4.1; [6] Theorem 3.4.5); for the first we need the Hahn-Banach theorem. If $T : X \to Y$ is not almost open then the dual $T^\dagger : Y^\dagger \to X^\dagger$ is not bounded below, and hence there is $g = (g_n)$ in $Y^\dagger$ for which

$$||g_n|| = 1 \text{ and } ||g_nT|| \to 0, \tag{8.3}$$

and then $y = (y_n)$ in Y for which

$$||y_n|| = 1 \text{ and } |g_n(y_n)| \geq \tfrac{1}{2} : \tag{8.4}$$

we claim that there is implication

$$x \in \ell_\infty(X) \Longrightarrow \mathrm{dist}(y - Tx, c_0(Y)) \geq \tfrac{1}{2}. \tag{8.5}$$

Indeed if $x = (x_n)$ is bounded then

$$||y_n - Tx_n|| = ||g_n||||y_n - Tx_n|| \geq |g_n(y_n - Tx_n)| \geq \tfrac{1}{2} - ||g_nT||||x_n|| \to \tfrac{1}{2},$$

and hence $\mathrm{limsup}_n||y_n - Tx_n|| \geq \frac{1}{2}$. This proves (8.2), and hence (8.1), if we note that the left-hand side is the condition that $\mathbf{Q}(T)$ is onto. □

The bounded closure of the range intervenes ([4] Theorem 11.3.2(c)) in the theory of compact operators:

9. Theorem. *If T is upper semi-Fredholm in the sense that*

$$x \in \ell_\infty(X), Tx \in m_1(Y) \Longrightarrow x \in m_1(X), \tag{9.1}$$

where $m_1(X)$ is the space of sequences $x \in \ell_\infty(X)$ of which every subsequence has a convergent subsequence, then

$$T(X) = \mathrm{cl}^\sim(T, X) = \mathrm{cl}\ T(X). \tag{9.2}$$

Proof. To verify the second equality in (9.2) we may by Theorem 2 replace T by its one-one part $T^\wedge : X/T^{-1}(0) \to Y$, and hence assume that T is one-one: we need to check that if (9.1) holds for T then it also holds for $T^\wedge$. We claim that if $||y - Tx_n|| \to 0$ then the condition (9.1) forces (x_n) to be bounded: for if $x' = (x'_n)$ were a subsequence of $x = (x_n)$ for which $||x'_n|| \to \infty$ then

$$||T(\frac{x'_n}{||x'_n||})|| = \frac{||Tx'_n||}{||x'_n||} \to 0 \tag{9.3}$$

and hence by (9.1) there would be a subsequence $x'' = (x''_n)$ of x' and an element $z_\infty \in X$ for which

$$||\frac{x''_n}{||x''_n||} - z_\infty|| \to 0. \tag{9.4}$$

But now $||Tz_\infty|| = 0$ while $||z_\infty|| = 1$, a contradiction.

To verify the first equality in (9.2) suppose $y = \mathrm{lim}_n Tx_n$ with bounded $x = (x_n)$: then

$$Tx \in c_1(Y) \subseteq m_1(Y), \tag{9.5}$$

and hence by (9.1)

$$x \in m_1(X). \tag{9.6}$$

Now if $x' \prec x$ is a convergent subsequence of x, with $x'_n \to x'_\infty \in X$, then

$$y = \mathrm{lim}_n Tx'_n = Tx'_\infty \in T(X).$$

□

The "upper semi-Fredholm condition" (9.1) agrees with more traditional conditions ([3] Theorem 1.3.2; [2] Theorem 2), even for incomplete spaces ([6] Theorem 6.9.2): indeed when (9.1) holds then (1.2) can be replaced by the stronger condition, that T is relatively open.

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