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FACTORIZATION OF HOLOMORPHIC MAPPINGS ON $C(K)$–SPACES

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Abstract. We prove a universal mapping theorem for a large class of holomorphic mappings F on a $C(K)$–space, stating that F can be locally written in the form $F(f) = B\big(1/(1 - Af)\big)$, where A and B are bounded linear operators on certain Banach spaces consisting of functions on K, and the division is taken pointwise.

Introduction

We prove a linearization theorem for a class of holomorphic mappings F on $C(K)$–spaces. We show in Theorem 3.4 that such an F can be presented as a compose of bounded linear operators A, B and the holomorphic mapping $H(f)(t) := 1/(1 - f(t))$, where $f \in U$ (the open unit ball of $C(K)$) and $t \in K$:

$$F(f) = BH(Af) = B\Big(\frac{1}{1 - Af}\Big). \tag{0.1}$$

Here only the operator B depends on F so that this result can be considered as a universal mapping theorem where both the universal map ($= H \circ A$) and also the universal space are of a very special form. The point is that the non–linearity of the universal map comes only from the simple scalar holomorphic map $z \mapsto 1/(1 - z)$.

Our result is only local: it deals only with mappings F defined on open discs. Moreover, there are some unsolved problems concerning the operators A and B. We refer to Theorem 3.4 and Remark 3.5.

Universal mapping theorems for holomorphic mappings on Banach or locally convex spaces have previously been studied for example in [Ma, Mu1, Mu2, Mu–N, G–G–M], see also [R].

1. Notation. Integral holomorphic mappings

We denote by $\mathbf{N}$ the set $\{1, 2, 3, \dots\}$ and by $\mathbf{N}_0$ the set $\mathbf{N} \cup \{0\}$. The closed unit interval $[0, 1]$ is denoted by I. All Banach spaces are over the complex scalar field. The space of bounded linear operators between the Banach spaces X and Y is denoted by $L(X, Y)$, or by $L(X)$, if $X = Y$; the dual of X is denoted by X^*. The absolutely convex hull of a subset A of a Banach space is denoted by $\Gamma(A)$.

For general topology we refer to [Ku]. If K is a compact metric space, we denote by $C(K)$ (resp. $\ell_\infty(K)$) the Banach space of continuous (resp. bounded), complex

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valued mappings $K \to \mathbf{C}$, endowed with the sup–norm. If K_1 and K_2 are compact metric spaces and $\varphi : K_1 \to K_2$ is a continuous surjection, we denote by φ° the linear isometry from $C(K_2)$ into $C(K_1)$ given by $\varphi^\circ f = f \circ \varphi$. If $\varphi^\circ(C(K_2))$ is 1–complemented in $C(K_1)$, i.e., if there exists a contractive projection from $C(K_1)$ onto $\varphi^\circ(C(K_2))$, we say that φ admits a regular averaging operator. (Note that in this case the map φ° also has a contractive left inverse.) For more details we recommend the reference [LT], Sections II.4.h,i, and [P].

For complex analysis in infinite dimensional spaces we refer to [D2] and [C]. If X and Y are Banach spaces and $n \in \mathbf{N}$, we denote by $\mathrm{P}({}^nX, Y)$ the space of continuous n–homogeneous polynomials $X \to Y$.

Recall that a continuous n–linear form F on $C(K)^n$ is called integral, if there exists a $\mu(F) \in C(K^n)^*$ such that

$$F(f_1, \dots, f_n) = \langle \prod_{k=1}^{n} f_k \circ \pi_n^{(k)}, \mu(F) \rangle, \tag{1.1}$$

where $f_k \in C(K)$, $\pi_n^{(k)}$ is the canonical projection from K^n onto the k:th coordinate space and the product on the right-hand side is taken pointwise.

Let $U \subset C(K)$ be open and $F : U \to Y$ holomorphic. We write the Taylor series of F at the point $y \in U$ as

$$F(x) = F_0 + \sum_{n=1}^{\infty} F_n^{(y)}(x - y), \tag{1.2}$$

where $F_0 \in Y$ and $F_n^{(y)} \in \mathrm{P}({}^nC(K), Y)$; we denote by $\hat{F}_n^{(y)}$ the corresponding symmetric n–linear mapping.

The following definition was given in [T2].

1.1. Definition. Let Y be a Banach space, let $U \subset C(K)$ be open, let $F : U \to Y$ be a holomorphic mapping, let $B \subset U$ be an open ball with center y and radius r, and let $S \subset Y^*$ be a bounded subset. We say that F is uniformly (S, B)–integral, if

1°. for every $t \in S$, $n \in \mathbf{N}$, the n–linear form

$$(f_1, \dots, f_n) \mapsto \langle \hat{F}_n^{(y)}(f_1, \dots, f_n), t \rangle$$

is integral (write $\mu(F, n, t)$ for the corresponding element of $C(K^n)^*$ as in (1.1)),

2°.

$$||F||_{S,B} := \sup_{t \in S} \Big\{ |\langle F_0, t \rangle| + \sum_{n=1}^{\infty} \sup_{\substack{h \in C(K^n), \\ ||h|| \le 1}} |\langle h, \mu(F, n, t) \rangle| r^n \Big\} < \infty, \tag{1.3}$$

and

3°. the mapping

$$t \to \sum_{n=1}^{\infty} \langle h_n, \mu(F, n, t) \rangle r^n \tag{1.4}$$

is, for arbitrary $h_n \in C(K^n)$ with $||h_n|| \le 1$, continuous $S \to \mathbf{C}$, when S is endowed with the weak* topology.

We remark that this concept of integral holomorphic mappings does not coincide with the definition of mappings of integral holomorphy type in [D1] and [A]. Nevertheless, the definition is quite natural and gives quite a large class of holomorphic mappings.

1.2. Examples. 1° The operator $f \mapsto f^n$ (pointwise multiplication; $n \in \mathbf{N}$) is uniformly integral $C(K) \to C(K)$ for every S and B as in Definition 1.1. We especially see that the identity operator on $C(K)$ is uniformly integral. (We refer to [T2] for the details of this and the following examples.)

2° Let $U \subset C(K)$ be the open unit ball, let $Y = C(K)$ and let h be a scalar valued holomorphic mapping on the open unit disc of $\mathbf{C}$ such that its Taylor coefficients at 0 form an absolutely summable sequence. Then the map $(Hf)(t) := h(f(t))$, $f \in U$, $t \in K$, is uniformly (K, U)–integral on U; here the set K is identified in the canonical way with a subset of $C(K)^*$.

3° Denote by $U \subset C(I)$ the open unit ball. If $F_n : I \times I^n \to \mathbf{C}$ is for all $n \in \mathbf{N}_0$ a continuous function satisfying $\sum_{n=0}^{\infty} ||F_n||_{C(I^{n+1})} < \infty$, then the holomorphic integral operator

$$f \mapsto \sum_{n=0}^{\infty} \int_{I^n} F_n(\cdot, s_1, \dots, s_n) f(s_1) \dots f(s_n) ds,$$

where $s = (s_1, \dots, s_n)$, is uniformly (I, U)–integral $U \to C(I)$.

2. Preliminary results

In this section we present some results necessary for the proof of the main result.

In the following universal mapping theorem we denote by K a compact metric uncountable space and by U the open unit ball of $C(K)$. The set K is also considered as a subset of $C(K)^*$: for every $t \in K$ there corresponds the point evaluation $\delta_t : f \mapsto f(t)$, $f \in C(K)$. This identification is a homeomorphism, when $C(K)^*$ is endowed with the weak*–topology.

2.1. Theorem. *There exists a universal holomorphic mapping $\psi : U \to C(K)$ such that for every uniformly (K, U)–integral holomorphic $F : U \to C(K)$ there exists $B_F \in L(C(K))$ such that the following diagram commutes:*

$$\begin{tikzcd}
U \arrow[rr, "F"] \arrow[dr, "\psi"'] & & C(K) \\
& C(K) \arrow[ur, "B_F"'] &
\end{tikzcd}$$

This result was proved in [T2], Theorem 2.1.

For the proof of the main theorem of this paper we shall need the following

Remark. Having a look at the proof of Theorem 2.1 of [T2] (especially (2.2) there) one easily verifies that

$$\psi(f) = f_0 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \psi_n^{(k)} f \tag{2.1}$$

where f_0 is an element of $C(K)$, $\psi_n^{(k)} \in L(C(K))$ and $||\psi_n^{(k)}|| \leq 1$ for all n and k.

2.2. Lemma. *Let $n \in \mathbf{N}$, let J be a finite set and let $(x_j)_{j\in J}$ be a sequence of complex numbers satisfying $|x_j| < 1$. For all sequences of complex numbers $(\lambda_j)_{j\in J}$*

$$\sup_{t\in[0,1]}\Big|\sum_{j\in J}\lambda_j\frac{1}{1-\mathrm{e}^{i2t\pi}x_j}\Big| \geq \Big|\sum_{j\in J}\lambda_j x_j^n\Big|. \tag{2.2}$$

Proof. Using the Taylor series of the analytic function $z \mapsto 1/(1-z)$, $|z| < 1$, we easily get

$$\int_0^1 \mathrm{e}^{-i2nt\pi}\sum_{j\in J}\lambda_j\frac{1}{1-\mathrm{e}^{i2t\pi}x_j}dt = \sum_{j\in J}\lambda_j x_j^n$$

This implies the inequality (2.2). □

2.3. Proposition. *There exist a strictly increasing sequence $(\tau(t))_{t=0}^{\infty}$, $\tau(t) \in \mathbf{N}_0$, $\tau(0) = 1$, and, for every $m \in \mathbf{N}$, $n \in \mathbf{N}_0$, complex numbers $a_{m,n}$ and $b_{m,n}$ such that the following holds (convention: $0^0 = 1$).*

For every $m \in \mathbf{N}$, $n \in \mathbf{N}_0$, we have $|a_{m,n}| < 1$, $|b_{m,n}| \leq e^7$.

For all $z \in \mathbf{C}$, $|z| < 1$, for all $n \in \mathbf{N}_0$,

$$\sum_{t\in\mathbf{N}_0}\sum_{m=\tau(t)}^{\tau(t+1)-1}\sum_{k\in\mathbf{N}_0} b_{m,n}a_{m,n}^k z^k = z^n, \tag{2.3}$$

and

$$\sum_{t\in\mathbf{N}_0}\sum_{m=\tau(t)}^{\tau(t+1)-1} b_{m,0} = 1. \tag{2.4}$$

This result is contained in Theorem 9 of [T4].

3. HOLOMORPHIC MAPPING AS A COMPOSE OF LINEAR OPERATORS AND A SCALAR HOLOMORPHIC FUNCTION

Let $U \subset C(K)$ be the open unit ball and let $r > \mathrm{e}$. In this section we show that a uniformly (K, rU)–integral holomorphic $F : rU \to C(K)$ can be presented as a product of linear operators A, B and the mapping $H : f(t) \mapsto 1/(1-f(t))$, $f \in U$, $t \in K$. More precisely, we show that the equality

$$F(f) = BH(Af) \tag{3.1}$$

holds for $f \in U$.

There are two major difficulties. First, one needs to solve at least approximately an infinite system of polynomially nonlinear equations. The solution is presented in detail in the paper [T4] and only the result is mentioned here; see Proposition 2.3. The second difficulty is to make the operators A and B well defined and continuous. We are not able to solve this problem in the optimal way. Accordingly, A becomes a bounded operator in the sup-norm, but the functions Af, where $f \in C(K)$, need not be continuous everywhere in K. (However, the discontinuity is in some sense only “mild”.) We are in general able to define the operator B only in the closed linear span of $H \circ A$, not in the whole space $C(K)$ (or $\ell_\infty(K)$). Finally, we need to assume that the given map F is holomorphic in rU, not only in U (see above). We refer to Remark 3.5 for some explanations.

3.1. Definitions. We choose for every $n, j \in \mathbf{N}_0$, $1 \le j \le 2^n$, a closed subinterval $I_{n,j} \subset I = [0,1]$ such that $I_{n,j} \cap I_{n',j'} = \emptyset$, if $n \ne n'$ or $j \ne j'$, and such that for every $\varepsilon > 0$ the interval $[0, 1-\varepsilon]$ contains only finitely many intervals $I_{n,j}$.

We choose for every n, j a continuous surjection $\varphi_{n,j} : I_{n,j} \to I^3$ with a regular averaging operator. (See [H], Theorem 2.2., also [T3], Theorem 2.) We denote by $\hat{\varphi}_{n,j}$ a contractive left inverse of $\varphi_{n,j}^\circ$.

We fix for every n, j a Borsuk–Kakutani extension operator $E_{n,j} : C(I_{n,j}) \to C(I)$ such that $\mathrm{supp}E_{n,j}f \cap \mathrm{supp}E_{n',j'}g = \emptyset$ for all f, g, if $n \ne n'$ or $j \ne j'$. (See [LT], Theorem II.4.14.; since the sets $I_{n,j}$ have mutually disjoint open neighbourhoods for different indices n, j, a simple trick shows that our requirement for the disjointness of the sets $\mathrm{supp}E_{n,j}f$ can be satisfied.) We fix some disjoint closed intervals $J_{n,j} \supset I_{n,j}$, $J_{n,j} \subset I$, such that $\mathrm{supp}E_{n,j}f \subset J_{n,j}$ for all $f \in C(I_{n,j})$.

For every $m \in \mathbf{N}$ and n, j we denote by $K_{m,n,j}$ a subspace of I^3 of the form $I_{n,j} \times \{s_2\} \times \{s_3\}$, where the numbers $s_2, s_3 \in I$ are chosen such that

$$s_2 e^{i2\pi s_3} = a_{m,n}. \tag{3.2}$$

Here $a_{m,n}$ is as in Proposition 2.3. We denote by $\eta_{m,n,j} : I_{n,j} \to K_{m,n,j}$ the homeomorphism $t \mapsto t \times \{s_2\} \times \{s_3\}$.

3.2. Lemma. *Use the notation of Proposition 2.3 and Definition 3.1, and fix some indices $n \in \mathbf{N}_0$, $j \in \mathbf{N}$, $1 \le j \le 2^n$. Let $\alpha \in L(C(I^3))$ be a contraction. Define $\alpha_n^{(j)} \in L(C(I^3))$ by*

$$(\alpha_n^{(j)} f)(t_1, t_2, t_3) = t_2 e^{i2\pi t_3} \big(E_{n,j} \varphi_{n,j}^\circ \alpha f\big)(t_1), \tag{3.3}$$

where $(t_1, t_2, t_3) \in I^3$.

1°. *We have $\|\alpha_n^{(j)} f\| \le \|\alpha f\|$ for all f.*

2°. *For every $f \in H(\alpha_n^{(j)}(U))$, where $U \subset C(I^3)$ is the open unit ball, the sum*

$$\beta_n^{(j)} f := \sum_{t=0}^{\infty} \sum_{m=\tau(t)}^{\tau(t+1)-1} b_{m,n} \hat{\varphi}_{n,j} \eta_{m,n,j}^\circ f \tag{3.4}$$

converges pointwise in I^3 and defines a continuous mapping from $H(\alpha_n^{(j)}(U))$ into $C(I^3)$ which can be extended as a bounded linear operator to the subspace $\overline{\mathrm{sp} H(\alpha_n^{(j)}(U))}$. Denoting the extension again by $\beta_n^{(j)}$ we have $\|\beta_n^{(j)}\| \le 1$.

3°. *We can define, without increasing the norm of $\beta_n^{(j)}$,*

$$\beta_n^{(j)} \sum_k \lambda_k H(f_k + g_k) = \beta_n^{(j)} \sum_k \lambda_k H(f_k) \tag{3.5}$$

for all finite sequences $(\lambda_k) \subset \mathbf{C}$, $(f_k) \subset \alpha_n^{(j)}(U)$ and $(g_k) \subset \ell_\infty(I^3)$ such that $\|g_k\| < 1$, $supp(g_k) \cap J_{n,j} \times I^2 = \emptyset$.

4°. *We have for $n \ge 1$*

$$\beta_n^{(j)} H(\alpha_n^{(j)} f) = (\alpha f)^n$$

for all $f \in U$, and $\beta_0^{(1)} H(0) = 1$.

Proof. 1°. This is clear.

2°. Assume that $f \in H(\alpha_n^{(j)}(U))$, $f = H(\alpha_n^{(j)} g)$ for some $g \in U$. Because of the definition of $K_{m,n,j}$ we have for every $m \in \mathbf{N}_0$

$$f \circ \eta_{m,n,j} = H(a_{m,n} \varphi_{n,j}^\circ \alpha g) = \varphi_{n,j}^\circ H(a_{m,n} \alpha g).$$

Hence,

$$\beta_n^{(j)} f = \sum_{t=0}^{\infty} \sum_{m=\tau(t)}^{\tau(t+1)-1} b_{m,n} H(a_{m,n}\alpha g) = \sum_{t=0}^{\infty} \sum_{m=\tau(t)}^{\tau(t+1)-1} b_{m,n} \sum_{k=0}^{\infty} (a_{m,n}\alpha g)^k, \tag{3.6}$$

and Proposition 2.3 implies the desired pointwise convergence of (3.4). Moreover, by Proposition 2.3 and (3.6),

$$\beta_n^{(j)} f = (\alpha g)^n, \tag{3.7}$$

so that $\beta_n^{(j)} f \in C(I^3)$.

We extend $\beta_n^{(j)}$ linearly to $\mathrm{sp}H(\alpha_n^{(j)}(U))$ and prove that the extension is a bounded operator. To this end let $J \subset \mathbf{N}$ be a finite sequence, let for every $k \in J$ the functions $f_k \in U$ and the complex numbers λ_k be arbitrary. We apply Lemma 2.2 to get the estimate

$$||\beta_n^{(j)} \sum_{k\in J} \lambda_k H(\alpha_n^{(j)} f_k)|| = || \sum_{k\in J} \lambda_k (\alpha f_k)^n ||$$

$$\le \sup_{t_2 \in I, t_3 \in I} \sup_{t \in I^3} \Big| \sum_{k\in J} \lambda_k \frac{1}{1 - t_2 e^{i2\pi t_3}(\alpha f_k)(t)} \Big|. \tag{3.8}$$

Recall that each $\varphi_{n,j}$ is a surjection and each $E_{n,j}$ is an extension operator. Hence, (3.8) is not greater than

$$\sup_{t_2 \in I, t_3 \in I} \sup_{t \in I} \Big| \sum_{k\in J} \lambda_k \frac{1}{1 - t_2 e^{i2\pi t_3}\big(E_{n,j}\varphi_{n,j}^{\circ} \alpha f_k\big)(t)} \Big|$$

$$= || \sum_{k\in J} \lambda_k H(\alpha_n^{(j)} f_k)||. \tag{3.9}$$

This proves the boundedness of $\beta_n^{(j)}$ in $E := \overline{\mathrm{sp}H(\alpha_n^{(j)}(U))}$ and the desired norm estimate.

3°. The operator $\beta_n^{(j)}$ is extended above to E. We have $\mathrm{supp} f \subset J_{n,j} \times I^2$ for all $f \in \alpha_n^{(j)}(C(I^3))$. Hence, for all $\lambda_k \in \mathbf{C}$, $f_k \in \alpha_n^{(j)}(U)$ and $g_k \in \ell_\infty(I^3)$ such that $||g_k|| < 1$ and $\mathrm{supp} g_k \cap J_{n,j} \times I^2 = \emptyset$,

$$||\beta_n^{(j)} \sum_k \lambda_k H(f_k + g_k)|| = ||\beta_n^{(j)} \sum_k \lambda_k H(f_k)||$$

$$\le || \sum_k \lambda_k H(f_k)|| \le || \sum_k \lambda_k H(f_k + g_k)||.$$

(The assumption on the supports is used to get the last inequality.)

4°. Follows from (3.7) and (2.4). □

3.3. Lemma. *Let* $((A_n^{(j)})_{j=1}^{2^n})_{n=1}^{\infty}$ *be a sequence of linear contractions* $C(I^3) \to C(I^3)$ *and let* $\varepsilon > 0$. *Let* $\psi : U \to C(I^3)$ *be the holomorphic mapping*

$$\psi(f)(t) := f_0(t) + \sum_{n=1}^{\infty} \sum_{j=1}^{2^n} \varepsilon_n^{(j)} (A_n^{(j)} f)(t)^n \tag{3.10}$$

where $\varepsilon_n^{(j)} \in \mathbf{C}$, $|\varepsilon_n^{(j)}| \le (2+\varepsilon)^{-n}$ *and* $f_0 \in C(I^3)$ *is fixed.*

There exist linear operators $A \in L(C(I^3), \ell_\infty(I^3))$ *and* $B_1 \in L(E, C(I^3))$, *where* $E \subset \ell_\infty(I^3)$ *is the closed linear span of* $H(A(U))$, *such that*

$$\psi(f) = B_1 H(Af) \tag{3.11}$$

for all $f \in U$.

Proof. We use the notations of Proposition 2.3, Definition 3.1 and Lemma 3.2. For every $n \in \mathbf{N}_0$ and $j = 1, \ldots, 2^n$ we choose the operators $\alpha_n^{(j)}$ and $\beta_n^{(j)}$ as in Lemma 3.2, taking $\alpha = A_n^{(j)}$ and $\alpha = 0$ in the case $n = 0$, $j = 1$. We define

$$A = \sum_{n,j} \alpha_n^{(j)},$$

$$B_1 = \beta_0^{(1)} + \sum_{n\in\mathbf{N}} \sum_{j=1}^{2^n} \varepsilon_n^{(j)} \beta_n^{(j)}. \tag{3.12}$$

That $A \in L(C(I^3), \ell_\infty(I^3))$ follows from 1° of Lemma 3.2 and from the assumption on the supports of the functions $E_{n,j}f$ (Definition 3.1). The boundedness of B_1 follows from the facts that $||\beta_n^{(j)}|| \leq 1$ and $|\varepsilon_n^{(j)}| \leq (2+\varepsilon)^{-n}$ for every n, j, and from 3° of Lemma 3.2. The statements 3° and 4° of Lemma 3.2 yield, for $f \in U$,

$$B_1 H(Af) = f_0 \beta_0^{(1)} H(0) + \sum_{n\in\mathbf{N}} \sum_{j=1}^{2^n} \varepsilon_n^{(j)} \beta_n^{(j)} H(\alpha_n^{(j)} f)$$

$$= f_0 + \sum_{n\in\mathbf{N}} \sum_{j=1}^{2^n} \varepsilon_n^{(j)} (A_n^{(j)} f)^n.$$

□

In the following theorem we denote by K a compact metrizable uncountable space which has a closed subspace homeomorphic to I and which is a Peano space (i.e. a continuous image of I), and by U the open unit ball of $C(K)$. Recall that for example every connected compact manifold is this kind of space K.

3.4. Theorem. *Let* $r > e$ *and let* $F : rU \to C(K)$ *be a uniformly* (K, rU)*–integral holomorphic mapping. There exist linear operators* $A \in L(C(K), \ell_\infty(K))$ *and* $B \in L(E, C(K))$, *where* $E \subset \ell_\infty(K)$ *is the closed linear span of* $H(A(U))$, *such that*

$$F(f) = BH(Af) = B\Big(\frac{1}{1 - Af}\Big) \tag{3.13}$$

for all $f \in U$.

Proof. 1°. We first consider the case $K = I^3$. One easily verifies that the mapping $G : U \to C(I^3)$, $G(x) := F(rx)$, is uniformly (K, U)–integral. We apply Theorem 2.1 to write $G = B_G \circ \psi_G$, where $B_G \in L(C(I^3))$,

$$\psi_G(f) = f_0 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \psi_n^{(k)} f \tag{3.14}$$

and $||\psi_n^{(k)}|| \leq 1$ for all n and k (see (2.1)). We get the representation $F = B_G \circ \psi$, where

$$\psi(f) = f_0 + \sum_{n=1}^{\infty} r^{-n} \prod_{k=1}^{n} \psi_n^{(k)} f. \tag{3.15}$$

For all $n \in \mathbf{N}$, the polarization formula ([D2], Theorem 1.5) implies the following equality for all complex numbers λ_k, $k = 1, \dots , n$:

$$\prod_{k=1}^{n} \lambda_k = \frac{1}{n!2^n} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \dots \varepsilon_n (\sum_{k=1}^{n} \varepsilon_k \lambda_k)^n .$$

Applying this we get

$$\psi(f) = f_0 + \sum_{n \in \mathbf{N}} r^{-n} 2^{-n} (n!)^{-1} n^n \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \dots \varepsilon_n \Big(n^{-1} \sum_{k=1}^{n} \varepsilon_k \psi_n^{(k)} f \Big)^n$$

$$= f_0 + \sum_{n \in \mathbf{N}} e^{-n} (n!)^{-1} n^n \sum_{\varepsilon_j = \pm 1} (2r/e)^{-n} \varepsilon_1 \dots \varepsilon_n \Big(n^{-1} \sum_{k=1}^{n} \varepsilon_k \psi_n^{(k)} f \Big)^n . \tag{3.16}$$

We have

$$e^{-n} (n!)^{-1} n^n \leq 1,$$

since $(n!)^{-1} n^n$ is the nth term in the Taylor series of e^n. Hence (3.16) yields a representation for ψ which satisfies the assumptions of Lemma 3.3; in particular, $A_n^{(j)} := n^{-1} \sum\limits_{k=1}^{n} \varepsilon_k \psi_n^{(k)}$. Let A and B_1 be as in Lemma 3.3. Setting

$$B = B_G B_1 \tag{3.17}$$

gives the the desired factorization of F in the case $K = I^3$.

2°. Let K be arbitrary. We first choose continuous surjections $\varphi : K \to I^3$ and $\varrho : I^3 \to K$ with regular averaging operators. Concerning ϱ, it is enough to take a retraction $I^3 \to I$ and compose it with a continuous surjection $I \to K$ having a regular averaging operator (see [H], Theorem 2.2., or [T3], Theorem 2.). The map φ can be found by composing a retraction from K onto a subspace homeomorphic to I (recall that I is an absolute retract space), with a continuous surjection $I \to I^3$ having a regular averaging operator. We denote a contractive left inverse of ϱ° by $\hat{\varrho}$. Taking a (usually discontinuous) right inverse φ^{-1} of φ one can define a contractive left inverse $\hat{\varphi}$ for φ° by $\varphi^{-1\circ} \in L(\ell_\infty(K), \ell_\infty(I^3))$.

If F is as in the assumption, it follows in a straightforward way from Definition 1.1 that $G := \varrho^\circ \circ F \circ \hat{\varrho}$ is a uniformly (I^3, V)–integral holomorphic mapping $V \to C(I^3)$, where $V \subset C(I^3)$ is the open unit ball. By part 1° of the proof we find bounded linear operators A_G and B_G as in (3.13) such that

$$G(f) = B_G H(A_G f)$$

for $f \in V$. We get for $f \in U$

$$F(f) = \hat{\varrho} \varrho^\circ F(\hat{\varrho} \varrho^\circ f) = \hat{\varrho} B_G H(A_G \varrho^\circ f)$$

$$= \hat{\varrho} B_G \hat{\varphi} \varphi^\circ H(A_G \varrho^\circ f) = \hat{\varrho} B_G \hat{\varphi} H(\varphi^\circ A_G \varrho^\circ f), \tag{3.18}$$

so that setting $B = \hat{\varrho} B_G \hat{\varphi}$ and $A := \varphi^\circ A_G \varrho^\circ$ yields the result. We leave the details to the reader. □

3.5. Remarks. 1°. In the case $K = I^3$ the elements of $A(C(K))$ can be discontinuous only in the subset $\{1\} \times I^2$ of the boundary of I^3. In the case of general K the discontinuity may be more serious; it depends on the choice of the mapping φ above.

The explanation for the discontinuity lies in the coefficients ε_k in the polarization formula (see (3.16)): they cause the space $A(C(K))$ to necessarily contain functions which oscillate infinitely often in K with a constant amplitude. The space $C(K)$ does not contain such elements.

2°. By the extension property of the space $\ell_\infty(K)$ it is always possible to extend B as a bounded operator $\ell_\infty(K) \to \ell_\infty(K)$. (See [LT1], Proposition 2.f.2.(iii).) In some cases it is possible to extend B even as a bounded operator $\overline{E + C(K)} \to C(K)$ where $\overline{E + C(K)}$ is considered as a closed subspace of $\ell_\infty(K)$; see Theorem 3.6 below.

3°. There is a natural explanation for the constant $r > e > 1$. It comes (modulo an $\varepsilon > 0$) basically from the fact that we cannot avoid the use of the polarization formula in (3.16). A related fact is that the "uniformly integral norm" (1.3) somehow measures the symmetric multilinear mappings in the Taylor series of the given F, whereas the representation (3.13) is more like a "polynomial of an infinite degree"; compare to [D2], Theorem 1.7.

3.6. Theorem. *Let r, K, U and F be as in Theorem 3.4. Assume that for every $n \in \mathbf{N}$ the linear operator*

$$(3.19) \quad T(F,n) \in L(C(K^n), C(K)) \ , \quad (T(F,n)f)(t) = \langle f, \mu(F,n,t)\rangle \quad \text{for } t \in K$$

where $\mu(F,n,t)$ is as in Definition 1.1, 1°, is compact. Then the operator B of Theorem 3.4 can be extended to an element of $L(\overline{E + C(K)}, C(K))$ or $L(\ell_\infty(K), C(K))$, where $\overline{E + C(K)}$ is considered as a closed subspace of $\ell_\infty(K)$.

Proof. The assumption (3.19) implies that B_F in the universal mapping Theorem 2.1 is compact. (The reader has to verify this from the proof of Theorem 2.1. of [T2], especially (2.4) and (2.5) there.) Hence, also the operator B is compact, see (3.12) and (3.17) etc. The result follows now from the extension properties of L_∞–spaces for compact operators, see [LT], Theorem II.5.25.2. ☐

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