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SOME APPLICATIONS OF FEJER'S THEOREM TO OPERATOR COSINE FUNCTIONS IN BANACH SPACES

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Abstract. We characterize spectral properties of operator cosine functions in Banach spaces in terms of the Césaro summability of two series associated to the resolvent of the corresponding infinitesimal generator.

1. Introduction

Given a strongly continuous cosine family of linear and bounded operators $(C(t))_{t\in\mathbf{R}}$ acting on a complex Banach space X with generator A, the following spectral inclusion

$$\cosh t\sqrt{\sigma(A)} \subset \sigma(C(t)), \quad t \in \mathbf{R},$$

was obtained by B. Nagy; he also proved that the reverse inclusion may fail quite drastically (see [9]). In particular $-N_0^2 \subset \rho(A)$ is implied by $1 \in \rho(C(2\pi))$ but not conversely.

In the case of a Hilbert space, I. Cioranescu and C. Lizama [1] proved the following result:

$$(*) \quad 1 \in \rho(C(2\pi)) \text{ if and only if } -N_0^2 \subset \rho(A) \text{ and } \underset{k\in\mathbf{Z}}{Sup}||kR(-k^2;A)|| < \infty;$$

however, the problem of the validity of $(*)$ in general Banach spaces was left open.

Results of this type for C_0- semigroups, as well as applications, were obtained by different authors (see [8], [6] and the references therein). Recently, G. Greiner and M. Schwarz [4] (see also G. Greiner [3]) have proved a spectral mapping theorem for C_0- semigroups in the Banach space setting involving the Césaro summability of the series $\sum_k R(ik;A)$. Their approach is based on the following vector valued version of the classical result due to Fejer [5].

Theorem 1.1. *Let X be a Banach space, $f : [0, 2\pi] \to X$ a continuous function and $z_k := \frac{1}{2\pi}\int_0^{2\pi} e^{-iks} f(s)ds$ its k-th Fourier coefficient; then the Fourier series of*

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f is Césaro summable to $f(t)$ in every point $t \in (0, 2\pi)$. Moreover, one has

$$C_1 - \sum_{k\in\mathbf{Z}} z_k := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} z_k = \frac{f(0)+f(2\pi)}{2}.$$

The aim of this work is to characterize the property "$1 \in \rho(C(2\pi))$" in terms of the Césaro summability of the series $\sum_k R(-k^2; A)$ and $\sum_k AR^2(-k^2; A)$. We also show, in section 3, that our main theorem reduces to $(*)$ in the Hilbert space case.

We remark that for operator cosine functions the behaviour of the single spectral value 1 of $C(2\pi)$ fails to characterize $\sigma(C(t))$ since the usual rescaling procedure, from the theory of C_0-semigroups (see [4], [8]), doesn't work.

Finally, in the last section, applications are made to prove the existence of solutions of some second order differential equations in Banach spaces with Dirichlet or Cauchy conditions.

2. Main result

The main result in this section is the following:

Theorem 2.1. *Let $(C(t))$ be an operator cosine function on the Banach space X, A its generator and $(S(t))$ the associated sine function; then the following assertions are equivalent:*

a) $1 \in \rho\,(C(2\pi))\,.$

b) $-N_0^2 \subset \rho(A)$ *and the sequences*

$$R_N = \tfrac{1}{N} \textstyle\sum_{n=0}^{N-1} \sum_{k=-n}^{n} R(-k^2; A) \quad \textit{and}$$

$$S_N = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} AR^2(-k^2; A)$$

are bounded in $\mathcal{L}(X)$.

c) $-N_0^2 \subset \rho(A)$ *and the limits*

$$Rx = \lim_{N\to\infty} R_N x \quad \textit{and} \quad Sx = \lim_{N\to\infty} S_N x$$

exist for every $x \in X$.

Proof. a) $\Rightarrow$ b) In [9] the following formula was proved:

$$\begin{aligned}(a^2 - A)\int_0^t &\sinh a(t-s)C(s)x\,ds \\ &= a[(\cosh at)x - C(t)x], \quad x \in X,\ a \in \mathbb{C},\ t \in \mathbf{R}.\end{aligned} \tag{1}$$

Integrating by parts and then taking the derivative with respect to t we obtain the following identity:

$$(a^2 - A)\int_0^t \cosh a(t-s)S(s)x\,ds = (\cosh at)x - C(t)x \tag{2}$$

for every $x \in X$ and $a \in \mathbb{C}$.

We now take $t = 2\pi$ and $a^2 = \lambda$ to obtain

$$(\lambda - A)\int_0^{2\pi} \cosh\sqrt{\lambda}(s-2\pi)S(s)x\,ds = (\cosh 2\pi\sqrt{\lambda})x - C(2\pi)x. \tag{3}$$

Assuming that $\cosh 2\pi\sqrt{\lambda} \in \rho\,(C(2\pi))$, we have

$$R(\lambda; A) = \left[\int_0^{2\pi} \cosh\sqrt{\lambda}(s-2\pi)S(s)ds\right][\cosh 2\pi\sqrt{\lambda} - C(2\pi)]^{-1}. \tag{4}$$

For the strongly continuous operator-valued function K on $[0,\ 2\pi]$, we denote by $\displaystyle\int_0^{2\pi} K(s)ds$ the linear and continuous operator defined by $\displaystyle\left(\int_0^{2\pi} K(s)ds\right)x := \int_0^{2\pi} K(s)xds,\ x \in X.$

From (4) one easily obtains

$$\begin{aligned} -R^2(\lambda; A) &= \frac{d}{d\lambda}R(\lambda; A) \\ &= \frac{1}{2\sqrt{\lambda}}\int_0^{2\pi}(s-2\pi)\sinh\sqrt{\lambda}(s-2\pi)S(s)[\cosh 2\pi\sqrt{\lambda} - C(2\pi)]^{-1}ds \\ &\quad - \int_0^{2\pi}\cosh\sqrt{\lambda}(s-2\pi)S(s)\frac{\pi\sinh 2\pi\sqrt{\lambda}}{\sqrt{\lambda}}[\cosh 2\pi\sqrt{\lambda} - C(2\pi)]^{-2}ds. \end{aligned} \tag{5}$$

If $1 \in \rho(C(2\pi))$, then by the spectral mapping theorem for operator cosine functions [9] we have $-N_0^2 \subset \rho(A)$. Then for $\lambda = -k^2$, $k \in \mathbb{Z}$, (4) and (5) respectively yield

$$R(-k^2; A) = \left(\int_0^{2\pi}\cos ksS(s)ds\right)[I - C(2\pi)]^{-1} \tag{6}$$

and

$$-R^2(-k^2; A) = \frac{1}{2k}\left(\int_0^{2\pi}(s-2\pi)\sin ksS(s)ds\right)[I - C(2\pi)]^{-1}.$$

It follows that (on $D(A)$ and therefore on all X) we have

$$\begin{aligned} -AR^2(-k^2; A) &= \frac{1}{2k}\left(\int_0^{2\pi}(s-2\pi)\sin ksAS(s)ds\right)[I - C(2\pi)]^{-1} \\ &= \frac{1}{2k}\left(\int_0^{2\pi}(s-2\pi)\sin ksC'(s)ds\right)[I - C(2\pi)]^{-1} \\ &= -\frac{1}{2k}\left[\int_0^{2\pi}k(s-2\pi)\cos ksC(s)ds + \int_0^{2\pi}\sin ksC(s)ds\right][I - C(2\pi)]^{-1} \\ &= -\frac{1}{2}\left[\int_0^{2\pi}(s-2\pi)\cos ksC(s)ds + \int_0^{2\pi}\frac{\sin ks}{k}S'(s)ds\right][I - C(2\pi)]^{-1} \\ &= -\frac{1}{2}\left[\int_0^{2\pi}(s-2\pi)\cos ksC(s)ds - \int_0^{2\pi}\cos ksS(s)ds\right][I - C(2\pi)]^{-1}. \end{aligned}$$

So that we finally obtain

$$AR^2(-k^2; A) = \frac{1}{2}\left(\int_0^{2\pi}\cos ks[(s-2\pi)C(s) - S(s)]ds\right)[I - C(2\pi)]^{-1}. \tag{7}$$

By (6) we have

$$\begin{aligned} R_N = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(-k^2;A) &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\int_0^{2\pi}\cos ks S(s)[I-C(2\pi)]^{-1}ds \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\int_0^{2\pi} e^{-iks}S(s)[I-C(2\pi)]^{-1}ds \end{aligned}$$

so that

$$R_N = \int_0^{2\pi}\sigma_N(s)S(s)[I-C(2\pi)]^{-1}ds \tag{8}$$

where $\sigma_N(s) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} e^{-iks} = \frac{1}{N}\frac{1-\cos Ns}{1-\cos s} \geq 0$ and $\int_0^{2\pi}\sigma_N(s)ds = 2\pi$.

Once again by (7) we obtain

$$\begin{aligned} S_N &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} AR^2(-k^2;A) \\ &= \int_0^{2\pi}\sigma_N(s)\frac{(s-2\pi)C(s)-S(s)}{2}[I-C(2\pi)]^{-1}ds. \end{aligned} \tag{9}$$

Now (8) and (9) yield

$$\|R_N\| \leq 2\pi M_1\|[I-C(2\pi)]^{-1}\| \text{ and } \|S_N\| \leq 2\pi M_2\|[I-C(2\pi)]^{-1}\|$$

with

$$M_1 = Sup\{\|S(s)\|; 0 \leq s \leq 2\pi\}$$

and

$$M_2 = Sup\left\{\frac{||(s-2\pi)C(s)-S(s)||}{2}; 0 \leq s \leq 2\pi\right\}.$$

b) ⇒ *c)* Let $-N_0^2 \subset \rho(A)$; then by the spectral mapping theorem for the residual spectrum for operator cosine functions (see [9]) it follows that $1 \notin \sigma_r(C(2\pi))$; this implies that $(I-C(2\pi))X$ is a dense subset of X. For $y \in X$, let $x = (I-C(2\pi))y$; then by the same arguments that led to (8) and (9) we have

$$\begin{aligned} R_N x &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(-k^2;A)(I-C(2\pi))y \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\int_0^{2\pi} e^{-iks}S(s)yds \end{aligned} \tag{10}$$

and

$$\begin{aligned} S_N x &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} AR^2(-k^2;A)(I-C(2\pi))y \\ &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\int_0^{2\pi} e^{-iks}\frac{(s-2\pi)C(s)-S(s)}{2}yds. \end{aligned} \tag{11}$$

By Fejer's theorem, it follows that the limits $Rx = \lim_{N\to\infty} R_N x$ and $Sx = \lim_{N\to\infty} S_N x$ exist for x in a dense subset of X. Since the sequences (R_N) and (S_N) are bounded in $\mathcal{L}(X)$, the limits exist for every $x \in X$.

c) $\Rightarrow$ *a)* Using again Fejer's theorem and letting $N \to \infty$, then from (10) and (11) we obtain

$$R(I - C(2\pi)) = \pi S(2\pi) \text{ and } 2S(I - C(2\pi)) = -2\pi^2 I - \pi S(2\pi).$$

Now, it follows that $(R + 2S)(I - C(2\pi)) = -2\pi^2 I$. Hence $-\frac{1}{2\pi^2}(R + 2S)$ is the inverse of $I - C(2\pi)$. (We notice that $R + 2S$ commutes with $I - C(2\pi)$.) □

In [2] the following class of operators was considered in connection with second order differential equations of elliptic type

Definition 2.2. A linear operator A defined in a Banach space X is called a-positive $(a > 0)$ if

$$\frac{-\pi^2 N_0^2}{a^2} \subset \rho(A) \text{ and } Sup_{k\in\mathbf{Z}} ||k^2 R(-\frac{\pi^2 k^2}{a^2}; A)|| < \infty.$$

Proposition 2.3. *If A is π-positive then the sequences*

$$R_N = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(-k^2; A) \quad \textit{and} \quad S_N = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} AR^2(-k^2; A)$$

are bounded in $\mathcal{L}(X)$.

Proof. We observe that by hypothesis there is a constant $M > 0$ such that

$$\|R(-k^2; A)\| \le \frac{M}{k^2}, \quad k \in \mathbf{Z}, \; k \ne 0,$$

from which the boundedness of $R_N = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{n=-n}^{n} R(-k^2; A)$ follows. On the other hand, we may write

$$S_N = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} AR^2(-k^2; A) = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(-k^2; A)AR(-k^2; A)$$

where $\|AR(-k^2; a)\| = \| - I + k^2 R(-k^2; A)\| \le 1 + M$. So we also obtain the boundedness of S_N. □

3. The Hilbert space case

We shall show that in the case of a Hilbert space we can derive from Theorem 2.1 the spectral mapping theorem from [1] (see also [7]) in which the summability conditions stated in (c) can be replaced by some boundedness conditions on the resolvents. For this purpose we shall need the following lemma which can be proved as for scalar-valued functions.

Lemma 3.1. *Let H be a Hilbert space. Assume $f : [0, 2\pi] \to H$ is continuous (square integrable). Then the Fourier coefficients $z_k := \frac{1}{2\pi}\int_0^{2\pi} f(s)e^{-iks}ds$ are square summable; more precisely,*

$$\sum_{k=-\infty}^{\infty} ||z_k||^2 = \frac{1}{2\pi}\int_0^{2\pi} ||f(s)||^2 ds.$$

Conversely, if $\{z_k\}$ *is a sequence in* H *and* $f_n := \sum_{k=-n}^{n} z_k e^{iks}$, *then*

$$\sum_{k=-n}^{n} ||z_k||^2 = \frac{1}{2\pi}\int_0^{2\pi} ||f_n(s)||^2 ds.$$

Theorem 3.2. *Let* $(C(t))$ *be an operator cosine function on the Hilbert space* H, A *its generator and* $(S(t))$ *the associated sine function. Then the following assertions are equivalent:*

(i) $1 \in \rho\,(C(2\pi))$.

(ii) $-N_0^2 \subset \rho(A)$ *and* $Sup_{k\in\mathbb{Z}}\|kR(-k^2;A)\| < \infty$.

Proof. (i)⇒(ii) It is an easy consequence of formula (1).

(ii)⇒(i) Suppose $\lambda \in \mathbb{C}$ is such that $\cos 2\pi\sqrt{\lambda} \in \rho\,(C(2\pi))$. Then replacing in formula (2) $t = 2\pi,\ a = \tau_k := \sqrt{\lambda} + ik$ and $T := [\cosh 2\pi\sqrt{\lambda} - C(2\pi)]^{-1}$ we have

$$\begin{aligned} R(\tau_k^2;A)x &= \left(\int_0^{2\pi} \cosh \tau_k(s-2\pi)S(s)ds\right)Tx \\ &= \frac{1}{2}\int_0^{2\pi} e^{iks}e^{\sqrt{\lambda}(s-2\pi)}S(s)Txds + \frac{1}{2}\int_0^{2\pi} e^{-iks}e^{-\sqrt{\lambda}(s-2\pi)}S(s)Txds \\ &= \int_0^{2\pi} e^{iks}F(s)xds + \int_0^{2\pi} e^{-iks}G(s)xds \end{aligned}$$

where $F(s) = \frac{1}{2}e^{\sqrt{\lambda}(s-2\pi)}S(s)T$, $G(s) = \frac{1}{2}e^{-\sqrt{\lambda}(s-2\pi)}S(s)T$ and x is fixed in H .

Then by Lemma 3.1, the sequence $\{u_k\} := \left\{R(\tau_k^2;A)x\right\}_{k\in\mathbb{Z}}$ is square summable.

We also have from (1) with $t = 2\pi,\ a = \tau_k$ and T as above that

$$\begin{aligned} \tau_k R(\tau_k^2;A)x &= \int_0^{2\pi} \sinh \tau_k(2\pi - s)C(s)Txds \\ &= \frac{1}{2}\int_0^{2\pi} e^{iks}e^{\sqrt{\lambda}(2\pi-s)}C(s)Txds - \frac{1}{2}\int_0^{2\pi} e^{-iks}e^{-\sqrt{\lambda}(2\pi-s)}C(s)Txds \\ &= \int_0^{2\pi} e^{iks}H(s)xds - \int_0^{2\pi} e^{-iks}L(s)xds \end{aligned}$$

where $H(s) = \frac{1}{2}e^{\sqrt{\lambda}(2\pi-s)}C(s)T$, $L(s) = \frac{1}{2}e^{-\sqrt{\lambda}(2\pi-s)}C(s)T$.

It follows that the sequence $\{v_k\} := \{\tau_k R(\tau_k^2;A)x\}_k$ is square summable.

Let $z_k := kR(\tau_k^2;A)x = i(\sqrt{\lambda}u_k - v_k)$; then it is clear that $\{z_k\}$ is also square summable.

From the resolvent equation we have

$$R(-k^2;A) = R(\tau_k^2;A)[I + (\lambda + 2ik\sqrt{\lambda})R(-k^2;A)] \tag{12}$$

so that $R(-k^2;A)x = [I + (\lambda + 2ik\sqrt{\lambda})R(-k^2;A)]u_k$ and $kR(-k^2;A)x = [I + (\lambda + 2ik\sqrt{\lambda})R(-k^2;A)]z_k$

Since by hypothesis $Sup_{k\in\mathbf{Z}}\ \|(\lambda + 2ik\sqrt{\lambda})R(-k^2;A)\| < \infty$, it follows that the sequences $\{R(-k^2;A)x\}$ and $\{kR(-k^2;A)x\}$ are square summable. Finally we get that the sequence $\{w_k\}_k$ where $w_k = (\lambda + 2ik\sqrt{\lambda})R(-k^2;A)x$ is square summable.

We shall prove that the conditions of the Theorem 2.1 (b) are satisfied.

Using (12) we have

$$\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(-k^2;A)x = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(\tau_k^2;A)x + \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(\tau_k^2;A)w_k.$$

We estimate the norm of the first term in the above sum:

$$\begin{aligned}\|\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} R(\tau_k^2;A)x\| &= \|\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n}\int_0^{2\pi}[e^{iks}F(s)x + e^{-iks}G(s)x]ds\| \\ &= ||\int_0^{2\pi}\sigma_N(s)(F(s)x + G(s)x)ds|| \\ &\le \int_0^{2\pi}\sigma_N(s)||F(s)x+G(s)x||ds \\ &\le Sup_{s\in[0,\ 2\pi]}\|F(s)x+G(s)x\|.\int_0^{2\pi}\sigma_N(s)ds < \infty.\end{aligned}$$

For the second term we have

$$\begin{aligned}\left\|\sum_{k=-n}^{n} R(\tau_k^2;A)w_k\right\| &= \left\|\sum_{k=-n}^{n}\int_0^{2\pi}(e^{iks}F(s)w_k + e^{-iks}G(s)w_k)ds\right\| \\ &\le \left(\int_0^{2\pi}\|F(s)\|^2ds\right)^{1/2}\left(\int_0^{2\pi}\|\sum_{k=-n}^{n}e^{iks}w_k\|^2ds\right)^{1/2} \\ &+ \left(\int_0^{2\pi}\|G(s)\|^2ds\right)^{1/2}\left(\int_0^{2\pi}\|\sum_{k=-n}^{n}e^{-iks}w_k\|^2ds\right)^{1/2}.\end{aligned}$$

Using Lemma 3.1 we obtain that the second term is also bounded.
From (12) we also have that

$$\begin{aligned}AR^2(-k^2;A) &= AR^2(\tau_k^2;A)[I+(\lambda+2ik\sqrt{\lambda})R(-k^2;A)]^2 \\ &= [-R(\tau_k^2;A)+\tau_k^2R^2(\tau_k^2;A)][I+(\lambda+2ik\sqrt{\lambda})R(-k^2;A)]^2 \\ &= -R(\tau_k^2;A) - R(\tau_k^2;A)[2(\lambda+ik\sqrt{\lambda})R(-k^2;A)+(\lambda+2ik\sqrt{\lambda})^2R^2(-k^2;A)] \\ &+ \tau_kR(\tau_k^2;A)\{\tau_kR(\tau_k^2;A)[I+(\lambda+2ik\sqrt{\lambda})R(-k^2;A)]^2\}.\end{aligned}$$

Denote

$$x_k = 2(\lambda+ik\sqrt{\lambda})R(-k^2;A)x + [(\lambda+2ik\sqrt{\lambda})R(-k^2;A)]^2x,$$

$$y_k = \tau_kR({\tau_k}^2;A)[I+(\lambda+2ik\sqrt{\lambda})R(-k^2;A)]^2x.$$

Then $\{x_k\}$ and $\{y_k\}$ are square summable.

We have

$$AR^2(-k^2;A)x = -R({\tau_k}^2;A)x - R({\tau_k}^2;A)x_k + \tau_k R({\tau_k}^2;A)y_k$$

so that

$$\frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=-n}^{n} AR^2(-k^2;A)x = -\frac{1}{N}\sum_{n=0}^{N-1}\sum_{-n}^{n} R({\tau_k}^2;A)x - \frac{1}{N}\sum_{0}^{N-1}\sum_{-n}^{n} R({\tau_k}^2;A)x_k$$

$$+\frac{1}{N}\sum_{0}^{N-1}\sum_{-n}^{n} \tau_k R({\tau_k}^2,A)y_k.$$

The first sum was estimated above; the second one can be exactly worked as was done previously and for the third one we have practically the same, since

$$\|\sum_{k=-n}^{n}\tau_k R(\tau_k^2;A)y_k\| = \|\sum_{k=-n}^{n}\int_0^{2\pi}(e^{iks}H(s)+e^{-iks}L(s))y_k ds\|$$

$$\leq \left(\int_0^{2\pi}||H(s)||^2 ds\right)^{1/2}\left(\int_0^{2\pi}\|\sum_{k=-n}^{n}e^{iks}y_k\|^2 ds\right)^{1/2}$$

$$+\left(\int_0^{2\pi}||L(s)||^2 ds\right)^{1/2}\left(\int_0^{2\pi}\|\sum_{k=-n}^{n}e^{-iks}y_k\|^2 ds\right)^{1/2} < \infty.$$

□

4. Applications

In this section, we first consider the following Dirichlet problem

$$\text{(4.1)}\qquad \begin{cases} u''(t) = Au(t), & 0 < t < \pi, \\ u(0) = x, & \\ u(\pi) = y, & \end{cases}$$

where $x, y \in D(A)$, A is a linear densely defined closed operator on a Banach space X and $u : (0,\pi) \to D(A)$ is a twice continuously differentiable function.

Theorem 4.1. *Let A be the generator of an operator cosine function $(C(t))$ on the Banach space X, and let $(S(t))$ be the associated sine function. Suppose that $-N_0^2 \subset \rho(A)$ and that both limits*

$$Rx = \lim_{N\to\infty} R_N x \quad \textit{and} \quad Sx = \lim_{N\to\infty} S_N x$$

exist for every $x \in X$.

Then, there is a unique solution of the Dirichlet Problem (4.1)*. Moreover, there is a constant c such that*

$$\text{(13)}\qquad Sup\{||u(s)|| : 0 \leq s \leq \pi\} \leq c(\|u(0)\| + \|u(\pi)\|).$$

Proof. From Theorem 2.1 it follows, under our hypothesis, that the operator $(I - C(2\pi))$ is invertible. Next, from the identity (see [10] Prop. 2.2)

$$C(t+s) - C(t-s) = 2AS(t)S(s)$$

taking $t = s = \pi$ we obtain that $S(\pi)$ is an invertible operator. Define

$$u(t) = S(t)S(\pi)^{-1}y + S(\pi - t)S(\pi)^{-1}x.$$

Then $u(t)$ is the (unique) solution of (4.1) and the result follows. $\square$

Corollary 4.2. *If A is π-positive and generates an operator cosine function, then the Dirichlet Problem* (4.1) *has a unique solution satisfying* (13).

Finally, we consider the second order abstract Cauchy problem

$$\text{(4.2)} \qquad \begin{cases} u''(t) = Au(t) + f(t), & t \in \mathbf{R}, \\ u(0) = x, \\ u'(0) = y. \end{cases}$$

We remark that if $f \in C^1(\mathbf{R}, X)$ and $(x, y) \in D(A) \times E$, where

$$E = \{x \in X / t \to C(t)x \text{ is once continuously differentiable } \},$$

then the classical solutions of (4.2) are given by the formula

$$u(t) = C(t)x + S(t)y + \int_0^t S(t-s)f(s)ds, \quad t \in \mathbf{R}. \tag{14}$$

If $(x, y) \in X \times X$ and $f \in L^2_{loc}(\mathbf{R}, X)$, we call (14) a *mild* solution of (4.2). It is clear that this mild solution is of class C^1 if and only if $(x, y) \in E \times X$.

The following result was proved in [1].

Theorem 4.3. *Let A be the generator of a strongly continuous cosine function $(C(t))$ in the Banach space X; then $1 \in \rho(C(2\pi))$ if and only if for any 2π-periodic function $f \in L^2_{loc}(\mathbf{R}, X)$, the equation* (4.2) *has a unique 2π-periodic mild solution of class C^1.*

Combining Theorems 2.1 and 4.3 we obtain the following

Corollary 4.4. *Let X be a Banach space and A the generator of a strongly continuous cosine function $(C(t))$ in X; the following assertions are equivalent:*

i) *The equation* (4.2) *has a unique 2π-periodic mild solution of class C^1 for any 2π - periodic function $f \in L^2_{loc}(\mathbf{R}, X)$.*

ii) *$-N_0^2 \subset \rho(A)$ and both limits*

$$Rx = \lim_{N\to\infty} R_N x \quad \textit{and} \quad Sx = \lim_{N\to\infty} S_N x$$

exist for every $x \in X$.

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