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THE JUNG THEOREM IN METRIC SPACES OF CURVATURE BOUNDED ABOVE

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ABSTRACT. The classical Jung Theorem states in essence that the diameter D of a compact set X in E^n satisfies $D \geq R[(2n+2)/n]^{1/2}$ where R is the circumradius of X. The theorem was extended recently to the hyperbolic and the spherical n-spaces. Here, the estimate above is extended to a class of metric spaces of curvature $\leq K$ introduced by A. D. Alexandrov. The class includes the Riemannian spaces. The extended estimate is of the form $D \geq f(R, K, n)$ where n is a positive integer suitably defined for the set X and its circumcenter. It can be that n is not unique or does not exist. In the latter case, no estimate is derived. In case of a Riemannian d-dimensional space, an integer n always exists and satisfies $n \leq d$. Then $D \geq f(R, K, n) \geq f(R, K, d)$. In case of E^d, one has $D \geq R[(2n+2)/n]^{1/2} \geq R[(2d+2)/d]^{1/2}$.

1. INTRODUCTION

The classical Jung Theorem [8, Theorem 2.6] states in essence that the diameter D of a compact set X in E^n satisfies

$$D \geq R\sqrt{(2n+2)/n}, \tag{1}$$

where R is the circumradius of X. The theorem was extended recently in two directions. First, estimate (1) was accompanied in [9] by a lower bound of the dimension of X in terms of the ratio D/R. Second, (1) was extended in [10] to the hyperbolic and the spherical n-space. Here, in the Theorem, we extend estimate (1) further, to a class of metric spaces introduced and studied first by A. D. Alexandrov [3, 4]. The class under our consideration includes Riemannian spaces; see Example (ii) below for details. This yields the Corollary dealing with the Riemannian case.

On Aug. 16, 1994, when the manuscript of this paper had already been finished, the author learned from an e-mail from Karoly Boroczky, Jr., that he was just one simple lemma short of proving a result probably similar to the Corollary. It is the author's pleasure to acknowledge this.

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To state the result exactly, we recall some definitions from [3, 4] and make a few remarks. Let M be an arbitrary metric space. Length of a curve in M, shortest

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paths and triangles in M are defined in the natural way as in [3, 4]. Each shortest path with the ends X, Y as well as its length will be denoted by XY. Let L_1, L_2 be two shortest paths in M coming from the same point O and let $X \in L_1$, $Y \in L_2$ be points different from O. Denote the distances OX, OY and XY by x, y and z respectively. (Note that a shortest path between X and Y does not necessarily exist.) Consider in E^2 a triangle with the side lengths x, y, z and let $\gamma(x,y)$ be its angle opposite to the side z. The number

$$\alpha = \limsup_{x,\,y\to 0} \gamma(x,y) \tag{2}$$

will be called the *upper angle* between L_1 and L_2. It obviously exists. Moreover, $\alpha \in [0, \pi]$. If there exists $\alpha = \lim_{x,\,y\to 0} \gamma(x,y)$, we will call it the *angle* between L_1 and L_2.

Denote by T a triangle in M with distinct vertices a, b, c and let α, β, γ be its upper angles at a, b, c respectively. The hyperbolic, Euclidean or spherical 2-dimensional space of curvature $K \in (-\infty, \infty)$ will be called the K-plane. Consider on the K-plane a triangle $a'b'c'$ satisfying $a'b' = ab$, $b'c' = bc$, $a'c' = ac$. Such a triangle always exists when $K \leq 0$. When $K > 0$, it exists under the additional assumption that its perimeter is less than $2\pi/\sqrt{K}$. Denote by α_K, β_K and γ_K its angles at the vertices a', b' and c'. The quantity

$$\delta_K(T) = (\alpha + \beta + \gamma) - (\alpha_K + \beta_K + \gamma_K) \tag{3}$$

is called the *excess of the triangle T with respect to K*.

Following [5, 3.1 and 1.2], we denote by R_K a metric space having the following properties :

(a) For any two points in R_K there exists a shortest path connecting these points. Its length is the distance between these points.
(b) The excess of any triangle in R_K with respect to K is non-positive.
(c) When $K > 0$, the perimeter of any triangle in R_K is less than $2\pi/\sqrt{K}$.

Remark 1. Alexandrov proved [4, p.39, §6, Theorem] that in R_K condition (b) implies that $\alpha - \alpha_K$, $\beta - \beta_K$, $\gamma - \gamma_K$ (see (3)) are each nonpositive. Moreover, the shortest path in R_K with prescribed ends is unique [4, p. 38].

The original definition of R_K in [4, p. 36] is somewhat different; see [5, 3.1., Remark]. It gives an impression that R_K is embedded into a bigger metric space and not necessarily connected. However it is practically used in [4] in the sense of our definition here. In the same sense R_K is understood in [6].

There is another widely studied class of metric spaces very close to R_K and denoted by P_K or CAT(K). A P_K-space by definition satisfies condition (a) and the following condition

(b′) every triangle in P_K of a perimeter less than $2\pi/\sqrt{K}$ (when $K > 0$) has a non-positive excess with respect to K.

Obviously each R_K is also a P_K. However a closed hemisphere of curvature K is a P_K but not an R_K. The class P_K was considered in [5, Theorems 5.3, 5.4, 6.4], [14], [15],[13], and [1]. (The notation for P_K in these papers is sometimes different.)

Remark 2. Return to the definition of the upper angle. The triangle with the side lengths x, y, and z from the definition can be constructed on the K-plane instead of E^2 for any K when x and y are sufficiently small. Let $\gamma^K(x,y)$ be the angle of such a triangle opposite to the side of the length z. An argument in [4, p. 35] shows that the upper angle α defined by (2) can be defined equivalently as follows:

$$\alpha = \limsup_{x,\, y\to 0} \gamma^K(x, y). \tag{4}$$

The same argument means that $\liminf \gamma(x, y) = \liminf \gamma^K(x, y)$ as $x, y \to 0$. Therefore the angle

$$\alpha = \lim_{x,\, y\to 0} \gamma(x, y), \tag{5}$$

if it exists, can be defined equivalently as

$$\alpha = \lim_{x,\, y\to 0} \gamma^K(x, y). \tag{6}$$

Remark 3. Let L_1, L_2 be two shortest paths in R_K coming from the same point O. By [4, p. 37], there exists an angle α between L_1 and L_2. Therefore the upper angles α, β, γ in (3) involved in the excess mentioned in (b) are actually angles.

We mention now four examples of R_K. The last three will be referred to.

(i) Let $[a, b]$, $a \neq b$, be a segment in E^1 with the standard metric in it. Then $[a, b]$ is an R_K for any K satisfying

$$K < \pi^2/(b - a)^2. \tag{7}$$

(Restriction (7) follows from condition (c).)

(ii) Let M^n, $n \geq 2$, be an n-dimensional Riemannian manifold and a closed set $C \subset M^n$ be convex, i.e. for each $a, b \in C$ there exists a unique shortest geodesic ab in M^n which lies in C. Suppose further that the sectional curvatures at each point of C are not greater than K and condition (c) holds in C. (It always holds when $\operatorname{diam} C < 2\pi/(3\sqrt{K})$.) Then, according to [4, p. 39, b)], the set C, as a subspace of M^n, is an R_K. (In fact, the word "unique" is missing from [4]. It was meant however; see Remark 1.)

(iii) Let $M_1 \in E^3$ consist of the xy-plane and the positive part of the z-axis. For $p, q \in M_1$, let the distance $\rho(p, q)$ be the length of the shortest path in M_1 between p and q. For instance,

$$\text{if } p = (1, 1, 0) \text{ and } q = (0, 0, 1), \text{ then } \rho(p, q) = \sqrt{2} + 1 \tag{8}$$

and the appropriate shortest path is the polygonal line pOq where O is the origin. One can check easily that each triangle T in the metric space M_1 has the excess $\delta_0(T) \leq 0$. Therefore M_1 is R_0. This follows also from Reshetnyak's gluing theorem [6, Theorem 6.1]. (A somewhat more general construction is described in [4, p. 38, b)] and [5, §4.1].)

(iv) Put

$$M_2 = \{(x, y) \mid y \leq 3|x|\}. \tag{9}$$

For $p, q \in M_2$, let the distance $\rho(p, q)$ be the length of the shortest path in M_2 between p and q. For instance, if $p = (-1, 0)$ and $q = (1, 3)$, then $\rho(p, q) = 1 + \sqrt{10}$ and the appropriate shortest path is the polygonal line pOq where O is the origin. One can check again that each triangle T in the metric space M_2 has the excess $\delta_0(T) \leq 0$. Therefore M_2 is R_0. (For more examples, see [5, §4]. See also the discussions in Gromov's papers [11, 12].)

Let c be a point in R_K. By a direction at c, we will mean a class of shortest paths coming from c each two of which subtend a zero angle. Denote by $\Omega = \Omega(c) =$

$\Omega(c, M)$ the set of all such directions at c. For $d_1,\ d_2 \in \Omega$, let $\angle(d_1, d_2)$ be the angle between a shortest path $ca_1 \in d_1$ and $ca_2 \in d_2$. This angle obviously does not depend on the choice of ca_1 and ca_2. The set Ω with the distance $\angle(.,.)$ becomes a metric space; see [4, p. 61, Theorem 2] and [5, §9] where the presentation however is more general.

Let for instance c be the origin in M_1 (see Example (iii)). Then Ω consists of a circle of the length 2π and a point at the distance π from each point of the circle. (Note that the angle between the z-axis and any horizontal ray coming from the origin equals π (not $\pi/2$) in the metric space M_1; see (6).) The set Ω at the origin in the space M_2 in Example (iv) is isometric to the segment $[-\arctan 3,\ \arctan 3]$ with the distance $\rho(x, y) = \min\{|x - y|,\ \pi\}$.

By the circumradius of a compact set X in a metric space M, we mean the number

$$R = \operatorname{circ} X \overset{\text{def}}{=} \inf_{q \in M} \max_{p \in X} \rho(p, q). \tag{10}$$

Each point $c \in M$ satysfying $\max_{p \in X} \rho(p, c) = R$ will be called a *circumcenter* of X. (It can be that the circumcenter is not unique or does not exist.) The closed ball of radius R centered at c will be called a *circumball*. For a point $p \in X \cap \partial B$, each shortest path cp will be called a *sensitive radius* of B and its direction $d \in \Omega(c)$ a *sensitive direction*.

Let sets $G,\ H \in \Omega(c)$. Suppose that $G \subset H$ and for each $h \in H$ there exists $g \in G$ such that $\angle(g, h) \geq \pi/2$. We will say then that G *blocks* H. For instance, if $H = \Omega(c) = S^1$, then the set G of three points with mutual distances $2\pi/3,\ 2\pi/3,\ 2\pi/3$ blocks H. The same is true when the distances are $\pi,\ \pi/2,\ \pi/2$. However when the distances are $\pi/2,\ \pi/4,\ \pi/4$, the appropriate set does not block H.

It will be convenient to unite estimate (1) with its analogs for the hyperbolic and spherical spaces in a single formula. For $n = 1, 2, 3, \ldots$, put

$$J_K^n(R) = \begin{cases} \frac{1}{\sqrt{-K}}\, 2\sinh^{-1}\left(\sqrt{\frac{n+1}{2n}}\sinh R\sqrt{-K}\right) & \text{for } K < 0,\ R \geq 0, \\ 2R\sqrt{\frac{n+1}{2n}} & \text{for } K = 0,\ R \geq 0, \\ \frac{1}{\sqrt{K}}\, 2\sin^{-1}\left(\sqrt{\frac{n+1}{2n}}\sin R\sqrt{K}\right) & \text{for } K > 0,\ R \in [0,\ \frac{\pi}{2\sqrt{K}}]. \end{cases} \tag{11}$$

Then the analog of (1) for the hyperbolic space (of curvature -1) is

$$D \geq J_{-1}^n(R), \quad R \geq 0; \tag{12}$$

see [10, (1.3)]. A simple rescaling argument (multiplication of the arc length element dS by a constant) turns (12) into

$$D \geq J_K^n(R), \quad K < 0,\ R \geq 0 \tag{13}$$

in the hyperbolic space of curvature $K < 0$.

Estimate (1) itself can be written as

$$D \geq J_K^n(R), \quad K = 0,\ R \geq 0. \tag{14}$$

The analog of (1) for the unit n-sphere under the restriction $R \leq \pi/2$ is

$$D \geq J_1^n(R), \quad R \in [0,\ \pi/2]; \tag{15}$$

see [10, (1.4)]. Due to the same rescaling argument, (15) turns into

$$D \geq J_K^n(R), \quad K > 0,\ R \in [0,\ \pi/(2\sqrt{K})], \tag{16}$$

in the n-sphere of curvature $K > 0$. The relations (13), (14) and (16) join now in the estimate

$$D \geq J_K^n(R), \quad \text{where } \begin{cases} R \geq 0 & \text{when } K \leq 0, \\ R \in [0, \pi/(2\sqrt{K})] & \text{when } K > 0. \end{cases} \tag{17}$$

2. The results

We are ready now to state our extension of (17) to a metric space R_K. Since (17) involves the dimension n, a surrogate of dimension of R_K will appear in our extension. It will be sufficient however to introduce and assume existence of such a surrogate only at a single point.

Theorem. *Let a set $X \in R_K$ be compact. Put $D = \operatorname{diam} X$, $R = \operatorname{circ} X$ and assume that*

$$R \leq \pi/(2\sqrt{K}) \quad \text{if } K > 0. \tag{18}$$

Suppose that

1. *A circumcenter c of X exists;*
2. *The circumball centered at c is compact.*

Denote by Σ the set of all sensitive directions at c. Then

(A) *Σ blocks $\Omega(c)$.*
(B) *If, for a positive integer n, the space $\Omega(c)$ has a subspace isometric to S^{n-1} and blocked by a subset of Σ, then* (17) *holds.*

Remark 4. Conditions 1 and 2 here obviously hold when R_K is compact. When $n = 1$, the set S^0 is a pair of points in $\Omega(c)$ at the distance π from each other.

Remark 5. If both S^{n-1} and $S^{m-1} \subset \Omega$ with $m > n$ are blocked by subsets of Σ, then (17) with n replaced by m can be disregarded since $J_K^n(R)$ decreases in n; see (11).

Remark 6. Applying the Theorem to E^n turns (17) into classical estimate (1) since in this case $\Omega = S^{n-1}$. The Theorem tells more however when $\Sigma \subset S^{k-1} \subset S^{n-1}$ for some $k < n$. Indeed, by the Theorem (A), the set Σ blocks $\Omega = S^{n-1}$. Since $\Sigma \subset S^{k-1}$, it blocks also S^{k-1}. Now, by (B), estimate (17) (and thus (1)) holds with n replaced by k. Note that condition (18) still allows conv X to be, say, n-dimensional. Similarly, the Theorem specifies estimates (12) and (15) proved in [10, (1.3), (1.4)].

Remark 7. Since the set $X \in R_K$ of the Theorem is compact, at least one sensitive direction d_1 obviously exists. It cannot be unique because, if it is, then the angle $\angle(d_1, d_1) \geq \pi/2$ by (A) while this angle should be zero by (6). Therefore, if $\Sigma \subset S^0$, then necessarily $\Sigma = S^0$, i.e. consists of two sensitive directions d_1 and d_2 with $\angle(d_1, d_2) = \pi$. Moreover, by (B), one has then $D \geq J_K^1(R) = 2R$; see (11). Since $D \leq 2R$, the equality $D = 2R$ holds when $\Sigma \subset S^0$.

Remark 8. Come back to the space M_1 in Example (iii). Denote by ρ_ϕ the point in the xy-plane with the polar coordinates $(1, \phi)$, $\phi \in [0, 2\pi)$. Consider the set $X_1 = \{p_0, p_\pi, p_{3\pi/2}\}$. Then $\operatorname{diam} X_1 = 2$, the circumball of X_1 is centered at the origin and $R = \operatorname{circ} X_1 = 1$. The set of sensitive directions here consists of three directions. Two of them corresponding to the points p_0 and p_π block a subspace

S^0 of the space Ω at the origin. (It was described before (10).) Then the Theorem (B) yields $\operatorname{diam} X_1 \geq 2R = 2$.

Consider now the set $X_2 = \{p_\varepsilon, p_\pi, p_{3\pi/2}\}$ for a small $\varepsilon > 0$. Now $\operatorname{diam} X_2$ is only a little less than 2. The circumball and the circumradius are the same. The sensitive directions however do not block any $S^0 \subset \Omega$. But they block $S^1 \subset \Omega$. Then the Theorem (B) yields a rougher estimate $\operatorname{diam} X_2 \geq R\sqrt{3} = \sqrt{3}$. Finally, put $X_3 = \{p_\varepsilon, p_\pi, p_{3\pi/2}, q\}$ where $q = (0,0,1)$. We have again $\operatorname{diam} X_3 = 2\,(= p_\varepsilon q)$. The circumball and the circumradius are the same. The sensitive directions corresponding to p_ε and q block another $S^0 \subset \Omega$. Again, the estimate is the best possible : $\operatorname{diam} X_3 \geq 2$.

For Riemannian spaces, the Theorem yields for instance the following.

Corollary. *Let M^r, $r \geq 2$, be an r-dimensional Riemannian space. Let C^r be a compact convex r-dimensional domain in it. (The convexity is understood as in Example (ii).) Suppose that the sectional curvatures in C^r are $\leq K$ and, when $K > 0$, the perimeter of any triangle in C^r is less than $2\pi/\sqrt{K}$. (This is always the case when $\operatorname{diam} C^r < 2\pi/(3\sqrt{K}$.) Let a set $X \subset \operatorname{int} C^r$ be compact. Put $D = \operatorname{diam} X$ and define $R = \operatorname{circ} X$ by* (10) *with M being the subspace C^r of M^r. Assume that* (18) *holds. Then*

- (A) *X, as a set in the metric space C^r, has a circumcenter c. It lies necessarily in $\operatorname{int} C^r$. The set Σ of the sensitive directions at c blocks the space S^{r-1} of all directions at c.*
- (B) *Suppose that, for some $n \in \{1, 2, \dots, r\}$, the space S^{r-1} has a subspace isometric to S^{n-1} and blocked by a subset of Σ. (This is always the case for $n = r$ due to (A).) Then* (17) *holds.*

Proof. According to Example (ii), the metric space C^r is an R_K. By compactness of C^r, the set X has a circumcenter c. Suppose $c \in \partial C^r$. Denote by Ω_i the set of all directions at c towards $\operatorname{int} C^r$. In terms of [7, §4.3], Ω_i is the intersection of the tangent cone of the set C^r at c with the standard unit sphere S^{r-1} in the tangent space $T_c M^r$. By [7, Theorem 4.3], this tangent cone is open and convex. Therefore Ω_i is an open convex set in S^{r-1}. The circumcenter d of such a set Ω_i lies in Ω_i and $\operatorname{circ} \Omega_i \leq \pi/2$. Since $\operatorname{int} C^r$ is open, the direction of any segment cx, $x \in \operatorname{int} C^r$, forms with d an angle $< \pi/2$. In particular this is true of any direction from the set Σ. Since $d \in \Omega_i \subset \Omega(c, C^r)$, the set Σ does not block $\Omega(c, C^r)$ contrary to the Theorem (A). Thus $c \in \operatorname{int} C^r$.

Hence the set $\Omega(c, C^r)$ is S^{r-1} and, by the Theorem (A), Σ blocks this S^{r-1}. The part (A) of the Corollary has now been proved. The part (B) follows from the Theorem (B). ∎

3. Proof of the Theorem

3.1. **Proof of part (A).** Suppose to the contrary that there is a direction $d \in \Omega$ which forms an angle $< \pi/2$ with each direction from Σ. To find a contradiction, we will construct a point b such that the set X fits into a ball centered at b and having a radius $< R$. To that end, we first split X into two parts as follows. Put

$$Y(\delta) = X \setminus \operatorname{int} B_c(R - \delta), \quad \delta \in [0,\, R), \tag{19}$$

where $B_c(r)$ is the closed ball of radius r centered at c. Put also

$$Z(\delta) = X \cap B_c(R - \delta). \tag{20}$$

Then

$$X = Y(\delta) \cup Z(\delta), \quad \delta \in [0, R). \tag{21}$$

Obviously $Y(\delta)$ and $Z(\delta)$ are compact and $Y(0)$ is non-empty.

Choose a shortest path cb of the length

$$cb < R/4 \tag{22}$$

in the direction d (which is a class of shortest paths). Let us show that there exists a number $\lambda > 0$ such that the angle

$$\angle acb < \pi/2 - \lambda \quad \text{for any } a \in Y(0). \tag{23}$$

(The angle is defined because the shortest path ac is unique by Remark 1.) By our contrary assumption, the angle $\angle acb < \pi/2$. If (23) fails, then there exists a sequence of the shortest paths $a_i c$, $a_i \in Y(0)$, such that the angle

$$\angle a_i cb \to \pi/2 \quad \text{as } i \to \infty. \tag{24}$$

The paths $a_i c$ lie in a compact domain: the circumball $B_c(R)$ (see condition 2 of the Theorem). Their lengths are all equal to R and thus uniformly bounded. Under these circumstances, Theorem 4 in [2, p. 62] guarantees a convergent subsequence in the sequence $a_i c$. (For a definition of the convergence, see [4, p. 54, bottom].) Therefore one may assume that the paths $a_i c$ converge to a curve qc. By Theorem 1 in [2, p. 67], the curve qc is a shortest path. Since $a_i \in Y(0)$ and by compactness of $Y(0)$, the point $q = \lim_{i\to\infty} a_i$ lies on $Y(0)$. Hence qc is a sensitive radius. By [4, p. 61, Theorem 3], the angle $\angle qcb = \lim_{i\to\infty} a_i cb$ and, by (24), $\angle qcb = \pi/2$. This is impossible by our contrary assumption. Hence (23) holds.

Having selected $\lambda > 0$, we choose now a suitable $\delta > 0$ in (19). For a point $p \in Y(\delta)$, denote by $a = a(p)$ a point in $Y(0)$ closest to p. Such a point exists by compactness of $Y(0)$. The shortest path pc is unique by Remark 1. Let us show that, for sufficiently small δ, the angle

$$\angle pca < \lambda/2. \tag{25}$$

If (25) fails, then there exist sequences $p_i \in Y(R/(2i))$ and $a_i = a_i(p_i) \in Y(0)$ such that

$$\angle p_i c a_i \geq \lambda/2, \quad i = 1, 2, \dots . \tag{26}$$

As above, one may assume that the paths $a_i c$ converge to a sensitive radius qc. Similarly, the paths $p_i c$ converge to a shortest path rc, where $r = \lim_{i\to\infty} p_i \in Y(0)$ (so that rc is also a sensitive radius). The points r and q should in fact coincide because otherwise, for large i, $p_i r < rq - p_i r - a_i q \leq p_i a_i$ contrary to the definition of $a_i(p_i)$. Then the radii cr and cq coincide and $\angle rcq = 0$. On the other hand, by [4, p. 61, Theorem 3], this angle equals $\lim_{i\to\infty} \angle p_i c a_i \geq \lambda/2$ by (26). Hence (25) holds.

There obviously exists a number $\varepsilon = \varepsilon(K, R, \lambda) > 0$ such that, for any triangle $p'c'b'$ in the K-plane with $p'c' \in [3R/4, R]$, $p'b' \geq p'c'$ and $c'b' < \varepsilon$, the angle

$$\angle p'c'b' > \pi/2 - \lambda/4. \tag{27}$$

We show now that, for any $p \in Y(\delta)$, the length pb satisfies

$$pb < R \tag{28}$$

when the distance $s \stackrel{\text{def}}{=} cb < \varepsilon$ and $\delta < R/4$ (in addition to the previous restrictions on cb and δ). Suppose to the contrary that $pb \geq R$. Then

$$pb \geq pc. \tag{29}$$

By the triangle inequality (in Ω) and (23), (25), the angle

$$\angle pcb \leq \angle acb + \angle pca < (\pi/2 - \lambda) + \lambda/2 = \pi/2 - \lambda/2. \tag{30}$$

Since $\delta < R/4$, one also has

$$3R/4 < pc \leq R. \tag{31}$$

Construct on the K-plane a triangle $p'c'b'$ with

$$p'c' = pc, \quad c'b' = cb, \quad p'b' = pb. \tag{32}$$

It exists in particular when $K > 0$ since $pc \leq R$, $cb < R/4$, see (22), and due to (18). By (31), (29) and (32), inequality (27) holds. With the points p and c fixed, the angle $\angle p'c'b'$ still depends on s. According to [3, §1.5, 1)] or [6, Proposition 5.2 and 1.4.2], there exists

$$\lim_{s \to 0} \angle p'c'b' = \angle pcb. \tag{33}$$

Now (27) implies that

$$\angle pcb \geq \pi/2 - \lambda/4.$$

This contradicts (30) which proves (28).

Due to (28), the set $Y(\delta) \subset \operatorname{int} B_b(R)$. Since $Z(\delta) \subset \operatorname{int} B_c(R)$, see (20), one has also $Z(\delta) \subset \operatorname{int} B_b(R)$ when cb is sufficiently small. Thus

$$X = Y(\delta) \cup Z(\delta) \subset \operatorname{int} B_b(R) \tag{34}$$

and hence $X \subset B_b(r)$ with $r < R$. This is the desired contradiction.

3.2. **Proof of part (B).** Let σ be a subset of Σ blocking S^{n-1}. Denote by Y the set of all points p from $X \cap \partial B_c(R)$ such that the direction of the radius cp is in σ. Let V^n_K be the hyperbolic, Euclidean, or spherical n-space of curvature K. (A line if $n = 1$.) Fix a point $\tilde{c}$ in V^n_K and identify the set of directions at $\tilde{c}$ with S^{n-1} mentioned above. For $p \in Y$, denote by $\tilde{p}$ the point in V^n_K such that the segment $\tilde{c}\tilde{p}$ has the length and the direction of cp. Note that $\tilde{p}$ exists and is unique since, in the spherical case, $cp = R \leq \pi/(2\sqrt{K})$. Moreover, the set

$$\tilde{Y} = \{\tilde{p} \,|\, p \in Y\} \tag{35}$$

lies on the boundary of the convex ball $B_{\tilde{c}}(R)$ of radius R centered at $\tilde{c}$.

Let us show that $B_{\tilde{c}}(R)$ is a circumball for $\tilde{Y}$.

Suppose to the contrary that $\tilde{Y}$ fits into a ball $B_a(r) \subset V^n_K$ with $r < R$. Obviously $a \neq c$. The $(n-1)$-dimensional circle $B_a(r) \cap \partial B_{\tilde{c}}(R)$ which contains $\tilde{Y}$ is smaller than a hemisphere of $\partial B_{\tilde{c}}(R)$. (For $n = 1$, this circle is just a point q.) Therefore the angle between the segment $\tilde{c}q$ with q being the center of that circle and any segment $\tilde{c}\tilde{p}$ with $\tilde{p} \in \tilde{Y}$ is $< \pi/2$. This is impossible since σ blocks S^{n-1}.

Due to (17),

$$\operatorname{diam} \tilde{Y} \geq J^n_K(R). \tag{36}$$

Fix an $\varepsilon > 0$ and let $\tilde{p}$, $\tilde{q} \in \tilde{Y}$ be such that the distance $\tilde{p}\tilde{q}$ in V^n_K satisfies

$$\tilde{p}\tilde{q} > \operatorname{diam} \tilde{Y} - \varepsilon. \tag{37}$$

(In fact there are $\tilde{p}$, $\tilde{q} \in \tilde{Y}$ such that $\tilde{p}\tilde{q} = \operatorname{diam} \tilde{Y}$ since $\tilde{Y}$ is compact. However we do not prove this compactness.)

The mapping $f : Y \to \tilde{Y}$ given by $f(p) = \tilde{p}$ is not necessarily one-to-one since the radii of $B_c(R)$ can branch (like the shortest paths qp_ε and qp_π in Remark 8). Let $p \in f^{-1}(\tilde{p})$, $q \in f^{-1}(\tilde{q})$. Consider a triangle $p'c'q'$ on the K-plane with $c'p' = c'q' = R$ and $p'q' = pq$. By the Theorem in [4, p. 39, §6],

$$\angle p'c'q' \geq \angle pcq = \angle \tilde{p}\tilde{c}\tilde{q}. \tag{38}$$

Comparison of the triangles $\tilde{p}\tilde{c}\tilde{q}$ and $p'c'q'$ yields

$$\tilde{p}\tilde{q} \leq p'q' = pq. \tag{39}$$

Due to (39), (37) and (36), one has

$$D = \operatorname{diam} X \geq pq \geq \tilde{p}\tilde{q} > \operatorname{diam} \tilde{Y} - \varepsilon \geq J_K^n(R) - \varepsilon. \tag{40}$$

Since (40) holds for any $\varepsilon > 0$, relation (17) holds in (B).

The Theorem has now been proved.

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