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COMMUTATIVITY CRITERIA FOR BANACH *-ALGEBRAS

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Abstract. Let A be a Banach *-algebra with an identity. Necessary and sufficient conditions are given for A to be commutative modulo its *-radical and for A to be commutative if A has a faithful *-representation as operators on a Hilbert space.

1. Introduction

This paper deals with a Banach *-algebra A with an identity. The positive cone P of A is an important item in its study. In [7] Kelley and Vaught used P in discussing *-representations of A as operators on a Hilbert space. More relevant to our present investigation is the use of P by Curtis [3] which we now recall. Let H be the set of all self-adjoint elements of A. A partial ordering of H is induced by P via the rule $h \geq k$ if and only if $h-k \in P$. Curtis showed that A is commutative if H is a lattice in that partial ordering. We take a different approach to commutativity using P. We show that A is commutative modulo its *-radical if and only if P is closed under Jordan multiplication (that is, $hk+kh \in P$ if $h, k \in P$). This result is used to help show the following. If A has a faithful *-representation, then A is commutative if and only if, given $u, v \in H$, there exists $w \in H$ where $u^2v^2 + v^2u^2 = w^2$. In particular this criterion holds if A is a semi-simple hermitian Banach *-algebra and so for a C^*-algebra.

2. Preliminaries

Throughout, A will denote a Banach *-algebra with an identity e. Let H be the set of self-adjoint elements of A. Let P_0 denote the set of all finite sums $\sum x_k^* x_k$ where each $x_k \in A$, and let Q_0 be the set of all finite sums $\sum h_j^2$ where each $h_j \in H$. Let $P(Q)$ denote the closure in H of $P_0(Q_0)$. In [3] and [7] the positive cone P is defined to be the closure of P_0 in A. But there the involution is assumed to be continuous. We take the closure of P_0 in H to take care of the case where the involution is not continuous.

As in [11] a linear functional $f(x)$ on A is said to be *positive* (*weakly positive*) if $f(x) \geq 0$ for all $x \in P_0(Q_0)$. As A has an identity each such $f(x)$ is real-valued on H. In [4, Ch. XV, sec. 7] a positive linear functional $f(x)$ is said to be a *trace* if $f(xy) = f(yx)$ for all $x, y \in A$. We have the corresponding notion of a *weak trace* for weakly positive linear functionals.

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A standard result [2, p. 198] is that any positive linear functional $f(x)$ on A is continuous on A (and thus $f(P) \geq 0$). This and the fact that a weakly positive linear functional on A is continuous are special cases of a neglected more general result [12, Th. 2.5] which reads as follows.

Theorem. *Let T be a linear mapping of A into a complex normed linear space. Suppose that there is an integer $n \geq 2$ and a real number $c > 0$ such that*

$$\|T(v^n + w^n)\| \geq c\|T(w^n)\|$$

for all $v, w \in H$. Then T is continuous.

We consider P and Q as closed cones in the real normed linear space H. It follows from Ford's square root lemma [5] that e is an interior point of Q and hence of P. Also (see [7, Lemma 1.2]) $P(Q) = H$ if and only if 0 is the only positive (weakly positive) linear functional on A.

Trivially $Q \subset P$. For the case of a Banach *-algebra B without an identity we have an example where $Q \neq P$. Let B be the algebra of all triples (λ, μ, ν) of complex numbers with the non-standard multiplication $(\lambda_1, \mu_1, \nu_1)(\lambda_2, \mu_2, \nu_2) = (0, 0, \lambda_1\mu_2 - \mu_1\lambda_2)$ and involution $(\lambda, \mu, \nu)^* = (-\overline{\lambda}, -\overline{\mu}, -\overline{\nu})$ where the norm $\|(\lambda, \mu, \nu)\| = (|\lambda|^2 + |\mu|^2 + |\nu|^2)^{1/2} B$ is a Banach *-algebra. Here $Q = (0)$ and P is the set of purely imaginary multiples of $(0, 0, 1)$. We do not have an example of A with identity where $P \neq Q$. We believe such exists but the following shows that an example may not be easily found. In §4 we shall need to have $Q = P$ in the cases there at hand.

Theorem 2.1. *Under any of the following conditions on A we have $P = Q$:* (1) *if A is commutative,* (2) *if A is a hermitian *-algebra,* (3) *if A is finite-dimensional,* (4) *if there exists some $x \in A$ where $x^*x = -e$.*

Proof. (1) If A is commutative, then $P_0 = Q_0$ so $P = Q$. (2) Let $x \in A$. By the Shirali-Ford Theorem ([2, p. 226]) we have $sp(x^*x) \subset [0, \infty)$. Take any $\varepsilon > 0$. Then $sp(\varepsilon e + x^*x) \subset (0, \infty)$. Hence, by [12, Lemma 2.4], there is some $w \in H$ where $\varepsilon e + x^*x = w^2$. It follows that $P = Q$. (3) If A is finite-dimensional, then every weakly positive linear functional is a positive linear functional, by [10, Cor. 7.4]. Then, by [7, Lemma 1.2] we see that $P = Q$. (4) Let $f(x)$ be a weakly positive linear functional. Calculations in [10, Lemma 7.2] show that $f(e) = 0$. As $|f(h)|^2 \leq f(e)f(h^2)$ for all $h \in H$ ([11, p. 231]) we see that $f(x) = 0$ for all $x \in A$. It follows that $Q = H$ so again $P = Q$.

An instance of (4) occurs for the algebra A of all two-by-two matrices over the complexes with a non-standard involution. We define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \overline{d} & \overline{b} \\ \overline{c} & \overline{a} \end{pmatrix}.$$

This is an involution on A and $x^*x = -e$ for

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. On cones and traces

We set forth our notation. Let Σ (Σ_w) be the set of all positive (weakly positive) linear functionals $f(x)$ on A where $f(e) = 1$. Let Φ denote the set of all non-zero multiplicative linear functionals $\gamma(x)$ on A where $\gamma(x^*) = \overline{\gamma(x)}$ for all $x \in A$. We consider the Jordan multiplication on A given by $x \cdot y = (xy + yx)/2$.

Note that $P(Q) = H$ if and only if Σ (Σ_w) is empty. In that case Φ is also empty. In the sequel we may suppose that $P(Q) \neq H$.

Lemma 3.1. *Each $\gamma \in \Phi$ is an extreme point of Σ (Σ_w).*

Proof. Suppose that $\gamma = af + (1-a)g$ where f and g are in Σ (Σ_w) and $0 < a < 1$. Let $h \in H$. Note that $f(h)^2 \leq f(h^2)$ and $g(h)^2 \leq g(h^2)$. Then

$$\gamma(h^2) = af(h^2) + (1-a)g(h^2) = \gamma(h)^2$$

so that

$$af(h^2) + (1-a)g(h^2) \leq a^2 f(h^2) + 2a(1-a)f(h)g(h) + (1-a)^2 g(h^2).$$

This implies that $f(h^2) + g(h^2) \leq f(h)g(h)$. Consequently $f(h)g(h) \geq 0$ for all $h \in H$.

We claim that $f(h) = g(h)$ for all $h \in H$ and so $f = g$. Suppose otherwise. Then there is some $v \in H$ where $f(v) \neq g(v)$, say $f(v) < g(v)$. Since $f(e) = g(e) = 1$ we can replace v by $w = te + v$ where t is real to obtain $f(w) < 0$ and $g(w) > 0$. But then $f(w)g(w) < 0$ which is impossible.

Theorem 3.1. *The following statements are equivalent.*

(a) *$P(Q)$ is closed under Jordan multiplication.*

(b) *Φ is the set of all extreme points of Σ (Σ_w)*

(c) *Every positive* (*weakly positive*) *linear functional is a trace* (*weak trace*).

Proof. Assume (a). To obtain (b) we must, using Lemma 3.1, show that each extreme point f_0 of Σ (Σ_w) is in Φ. Fix $y \in P(Q)$ where $e - y \in P(Q)$. We set

$$g_y(x) = f_0(x \cdot y) - f_0(x)f_0(y)$$

for all $x \in A$.

Clearly $g_y(e) = 0$. Next we set $w_1(x) = f_0(x) + g_y(x)$ and $w_2(x) = f_0(x) - g_y(x)$ and note that $w_1(e) = w_2(e) = 1$. Now we use (a).

For $x_0 \in P(Q)$ we have

$$w_1(x_0) = f_0(x_0)(1 - f_0(y)) + f_0(x_0 \cdot y) \geq 0.$$

Note that $x_0 \cdot (e - y) = x_0 - (x_0 \cdot y) \in P(Q)$. Then

$$w_2(x_0) = f_0(x_0 - (x_0 \cdot y)) + f_0(x_0)f_0(y) \geq 0.$$

Therefore w_1 and w_2 are in Σ (Σ_w). Inasmuch as $f_0 = (w_1 + w_2)/2$ and f_0 is an extreme point of Σ (Σ_w) we see that $g_y = 0$. Hence

$$f_0(x \cdot y) = f_0(x)f_0(y) \tag{1}$$

for all $x \in A$ and for all $y \in P(Q)$ where also $e - y \in P(Q)$. Now we consider an arbitrary $z \in P(Q)$. As e is in the interior of $P(Q)$ as a subset of H there is a real number $t > 0$ such that $e - tz \in P(Q)$ and $tz \in P(Q)$. Therefore, from (1), we see that (1) also holds for all $x \in A$ and $y \in P(Q)$. However, by [8, p. 208], $H = P - P = Q - Q$. Also $A = H + iH$ so that we see that (1) is valid for all $x, y \in A$.

A theorem of Jacobson and Rickart [6, Th. 2] asserts that a Jordan homomorphism of a ring into an integral domain is either a homomorphism or an antihomomorphism. It follows that f_0 is a multiplicative linear functional and thus lies in Φ.

Now we assume (b). Let $f \in \Sigma$ (Σ_w). By the Krein-Milman theorem f is the w^*-limit in A^* of elements of the form $\sum_{k=1}^n a_k\gamma_k$ where each $a_k \geq 0$, $\sum_{k=1}^n a_k = 1$ and each $\gamma_k \in \Phi$. As each γ_k is a trace (weak trace) it follows that so is f.

Next we assume (c). We treat the positive and weakly positive cases separately. Suppose first that every positive linear functional $f(x)$ is a trace. Let $w_1, w_2 \in P_0$. We show that $f(w_1 \cdot w_2) \geq 0$. We write $w_1 = \sum_{j=1}^n x_j^* x_j$ and $w_2 = \sum_{k=1}^m y_k^* y_k$. Then $2w_1 \cdot w_2$ is the finite sum of elements of the form $x^*xy^*y + y^*yx^*x$. As f is a trace we have

$$f(x^*xy^*y + y^*yx^*x) = 2f(x^*xy^*y) = 2f(yx^*xy^*) \geq 0.$$

Now let $v_1, v_2 \in P$. By the continuity of f we get $f(v_1 \cdot v_2) \geq 0$. As f is an arbitrary positive linear functional we see, by [7, Lemma 1.2], that $v_1 \cdot v_2 \in P$.

Next suppose that every weakly positive linear functional $g(x)$ is a weak trace and let $w_1, w_2 \in Q_0$. We write $w_1 = \sum_{j=1}^n h^2$ and $w_2 = \sum_{k=1}^m z_k^2$ where each h_j and z_k is in H. To see that $g(w_1 \cdot w_2) \geq 0$ it is enough to show that $g(h^2z^2 + z^2h^2) \geq 0$ for all $h, z \in H$. First note that

$$g(h^2z^2 + z^2h^2) = g(hz^2h + zh^2z).$$

However

$$(hz + zh)^2 + [i(hz - zh)]^2 = 2(hz^2h + zh^2z)$$

where $hz + zh \in H$ and $i(hz - zh) \in H$. Thus $hz^2h + zh^2z \in Q_0$ and so $g(h^2z^2 + z^2h^2) \geq 0$. We then see, by the argument used above for P, that Q is closed under Jordan multiplication. Hence (c) implies (a).

Corollary 3.1. *If Q is closed under Jordan multiplication, then $P = Q$.*

Proof. If Q is closed under Jordan multiplication, then, by Theorem 3.1, every weakly positive linear functional is a weak trace. Hence every positive linear functional is a trace. By Theorem 3.1, Φ is the set of extreme points of each of Σ and Σ_w. Hence $\Sigma = \Sigma_w$. Then, by [7, Lemma 1.2], we have $P = Q$.

4. Commutativity criteria

Theorem 4.1. *A is commutative modulo its *-radical if and only if P is closed under Jordan multiplication.*

Proof. Let P' denote the set of all positive linear functionals on A. By Theorem 3.1, P is closed under Jordan multiplication if and only if each $f \in P'$ is a trace. The latter is the case if and only if $xy - yx \in \bigcap\{f^{-1}(0)\colon\ f \in P'\}$ for all $x, y \in A$. But $\bigcap\{f^{-1}(0)\colon\ f \in P'\}$ is the *-radical of A; see [9, p. 265].

Corollary 4.1. *Suppose that A has a faithful *-representation as bounded linear operators on a Hilbert space. Then A is commutative if and only if for each pair $u, v \in H$ there exists $w \in H$ so that $u^2v^2 + v^2u^2 = w^2$.*

Proof. Suppose the condition on H holds. Let $x, y \in Q_0$ where $x = \sum_{j=1}^n h_j^2$ and $y = \sum_{k=1}^m z_k^2$. Then $2x \cdot y = \sum_j \sum_k h_j^2 \cdot z_k^2$. Therefore $x \cdot y$ is equal to the sum of squares of elements of H. This shows that Q_0 is closed under Jordan multiplication and then so is Q. By Corollary 3.1, P is closed under Jordan multiplication. Applying Theorem 4.1 we see that A is commutative.

Theorem 4.2. *A hermitian Banach* **-algebra* A *is commutative modulo its radical if and only if* $sp(u^2v^2 + v^2u^2) \subset [0, \infty)$ *for all* $u, v \in H$.

Proof. In this situation $Q = P$ by Theorem 2.1. Let $\Gamma = \{h \in H\colon\ sp(h) \subset [0, \infty)\}$. It is known, [9, (5.6)], that Γ is a cone in H. We show first that Γ is closed in H. By Ford's Lemma [5], if $\|e - h\| < 1$ for some $h \in H$, then $h = w^2$ for some $w \in H$. Consequently e lies in $\mathrm{Int}(\Gamma)$, the interior of Γ as a subset of H. Thus Γ is a convex set with interior. Let y_0 be in the closure of Γ in H. For each real number $t, 0 < t \leq 1$, we have $te + (1-t)y_0 \in \mathrm{Int}(\Gamma)$. Therefore $sp(te + (1-t)y_0) \subset [0, \infty)$ and $sp(y_0) \subset [-t(1-t)^{-1}, \infty)$ for each $t, 0 < t < 1$. Consequently $y_0 \in \Gamma$.

Clearly, [9, (5.6)], we have $Q_0 \subset \Gamma$, and, as Γ is closed in H, we see that $P \subset \Gamma$. We show next that $\Gamma \subset P$. Let $y \in \Gamma$ and $0 < t < 1$. Then $sp(te+(1-t)y) \subset (0, \infty)$ so that, by [12, Lemma 2.4], there exists $w \in H$ where $w^2 = te + (1-t)y$. Letting $t \to 0$ we see that $y \in Q$.

Now we suppose that $u^2v^2 + v^2u^2 \in \Gamma$ for all $u, v \in H$. Let $x = \sum_{j=1}^{n} h_j^2, y = \sum_{k=1}^{m} z_k^2$ be two elements in Q_0. Then $x \cdot y$ is the sum of a finite number of summands of the form $u^2v^2 + v^2u^2$ so $x \cdot y \in \Gamma = P$. We see that P is closed under Jordan multiplication. By Theorem 4.1, A is commutative modulo its *-radical. However here the *-radical coincides with the radical by [9, Th. 6.6]. We point out that Theorem 4.2 does not follow from Corollary 4.1. It is true that if A is a C^*-algebra and $w \in H$ with $sp(w) \subset [0, \infty)$, then $w = h^2$ for some $h \in H$. Easy examples show this is false for hermitian Banach *-algebras even though the required h exists if $sp(w) \subset (0, \infty)$; see [12, Lemma 2.4].

Corollary 4.2. *Let* A *be a semi-simple hermitian Banach* **-algebra. The following statements are equivalent:*

(a) A *is commutative.*
(b) $sp(u^2v^2 + v^2u^2) \subset [0, \infty)$ *for all* $u, v \in H$.
(c) *Given* $u, v \in H$ *there exists* $w \in H$ *so that* $u^2v^2 + v^2u^2 = w^2$.

Proof. Here the *-radical and the radical of A are both (0) and A has a faithful *-representation; see [9, (6.6)]. Then (c) is equivalent to (a) by Corollary 4.1 and (b) is equivalent to (a) by Theorem 4.2.

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