

A NOTE ON TCHAKALOFF'S THEOREM

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ABSTRACT. A classical result of Tchakaloff on the existence of exact quadrature formulae up to a given degree is extended to positive measures without compact support. A criterion for the existence of Gaussian quadratures for a class of such measures is also derived from the main proof.

1. INTRODUCTION

Tchakaloff's Theorem [9] asserts the existence of an exact quadrature formula with positive coefficients for polynomials of prescribed degree in n real variables and with respect to a positive, compactly supported measure which is absolutely continuous with respect to Lebesgue n -volume measure. This result was the source of several further developments in the theory of quadrature formulas; cf. [4], [5], [6] and [7].

The present note generalizes Tchakaloff's Theorem to arbitrary positive measures in the Euclidean space. The motivation for this investigation comes from some recent interesting work of Curto and Fialkow on truncated multivariable moment problems. One of the first natural questions when studying truncated moment problems is whether the existence of an arbitrary measure as solution to such a problem implies the existence of a solution equal to an atomic measure. It was exactly this question which was recently raised by Curto and Fialkow in [2] together with many other relevant open problems. The comments and the references in [2] give the reader an accurate description of the current status of the (still mysterious and intriguing) truncated multivariable moment problem.

Let $x = (x_1, \dots, x_n)$ be the current vector in \mathbf{R}^n and let $P_d(\mathbf{R}^n)$ be the real vector space of polynomials in x of total degree less or equal than d . Let $N_d(n)$ be the dimension of $P_d(\mathbf{R}^n)$. The main results of this note are contained in the following two theorems. Although the first result is not new, we state and prove it in this form as a preparation for Theorem 2. See for details [3]; [4], Section 2.4; [5], Chapter 7; and [7], Section 3.1.

Theorem 1. *Let μ be a positive measure with compact support K in \mathbf{R}^n and let d be a fixed positive integer.*

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Then there are N points ($N \leq N_d(n)$), $x_j \in K$, and positive real numbers c_j ($1 \leq j \leq N$) such that:

$$(1) \quad \int_{\mathbf{R}^n} p d\mu = \sum_{j=1}^N c_j p(x_j)$$

for all $p \in P_d(\mathbf{R}^n)$.

In other words, the measure μ and the linear combination of Dirac measures $\sum_{j=1}^N c_j \delta_{x_j}$ have the same moments up to degree d .

Theorem 2. *Let d be a positive integer and let μ be a positive measure in \mathbf{R}^n with the property that $\int |p| d\mu$ exists for every $p \in P_{2d}(\mathbf{R}^n)$. Then there are N points ($N \leq N_{2d}(n)$), $x_j \in \text{supp}(\mu)$, and positive real numbers c_j ($1 \leq j \leq N$) such that the quadrature relation (1) holds for all $p \in P_{2d-1}(\mathbf{R}^n)$.*

Above $\text{supp}(\mu)$ denotes the closed support of the measure μ . Notice that Theorem 2 does not guarantee the validity of the quadrature identity (1) in degree $2d$.

By restricting the support of the measure μ to a proper closed convex cone of \mathbf{R}^n , an adaptation of the proof of Theorem 2 will show that this result remains valid in any degree (even or odd). As a modest consequence of this observation we give at the end of the note a criterion for a measure with unbounded support to possess a Gaussian quadrature, that is, a quadrature with a minimal number of points.

2. PROOFS

The proof of Theorem 1 closely follows the principal ideas and convexity techniques of Tchakaloff [9]. First, we consider the vectors

$$v(x) = (1, x_1, x_2, \dots, x_n^d) \in \mathbf{R}^M$$

which enumerate in a prescribed order all monomials of total degree less than or equal to d . To simplify notation we put $M = N_d(n)$.

Let μ be a positive measure supported by the compact subset K of \mathbf{R}^n . Without loss of generality we can assume that $0 \in K$.

Let $F(K)$ be the vector space generated in \mathbf{R}^n by the vectors $v(x), x \in K$, and let N be the dimension of $F(K)$. Since the orthogonality relation

$$\langle v(x), q \rangle = 0 \quad (x \in K)$$

means that the polynomial $q(x) = \langle v(x), q \rangle$ vanishes identically on K , the dimension N less strictly than M reflects the fact that K is a compact subset of an algebraic variety.

Let us remark that there are N points $y_j \in K$, with the property that the vectors $v(y_j), 1 \leq j \leq N$, are linearly independent. Indeed, let N' be the largest number of linearly independent vectors among $v(y), y \in K$. Let $v(y_j), 1 \leq j \leq N'$, be such a system of independent vectors. Then every other vector $v(y)$ is linearly dependent of $v(y_1), \dots, v(y_{N'})$; that is, N' is necessarily equal to N .

Next we define the set

$$V(K) = \{v(x); x \in K\}.$$

Because K is compact it follows that $V(K)$ is compact in $F(K)$. Moreover, the preceding argument shows that $V(K)$ contains a basis of $F(K)$. Since $v(0) = (1, 0, 0, \dots, 0) \in V(K)$ and

$$\langle v(0), v(x) \rangle = 1 \quad (x \in K)$$

we infer that $V(K)$ is a fundamental subset of the Euclidean space $F(K)$, in the terminology of [6], Definition 5.3-4.

Following Tchakaloff [9] we define the convex cone $C_N(K)$ consisting of those elements of $F(K)$ which can be written as linear combinations with positive coefficients of at most N elements of $V(K)$. Theorem 3.3-4 of [6] shows then that $C_N(K)$ is a *closed* convex cone. Consequently Minkowski's Separation Theorem and an elementary linear algebra observation imply that, if a vector $u \in F(K)$ satisfies $\langle u, w \rangle \geq 0$ for all $w \in F(K)$ with the property that $\langle v(x), w \rangle \geq 0$ ($x \in K$), then $u \in C_N(K)$. See for details [6], Theorem 3.3-5.

Let $a = \int_K v(x)d\mu(x)$ be the vector of \mathbf{R}^M having as coordinate entries the moments of the measure μ , up to order d . It is obvious that $a \in F(K)$. Let $w \in F(K)$ be a vector satisfying $\langle w, v(x) \rangle \geq 0$ for all $x \in K$. Then

$$\langle w, a \rangle = \int_K \langle w, v(x) \rangle d\mu(x) \geq 0.$$

Therefore, by virtue of the above mentioned separation result $a \in C_N(K)$; that is, there are vectors $v(x_j), x_j \in K$, and non-negative coefficients c_j ($1 \leq j \leq N$) such that:

$$a = \sum_{j=1}^N c_j v(x_j).$$

In other words this is equivalent to the identity:

$$(2) \quad \int_K qd\mu = \langle q, a \rangle = \sum_{j=1}^N c_j q(x_j),$$

for every polynomial $q(x) = \langle q, v(x) \rangle$ corresponding to a vector $q \in F(K)$.

For an arbitrary vector $q \in P_d(\mathbf{R}^n)$ we write $q = q' + q''$ with $q' \in F(K)$ and $q'' \perp F(K)$. Then we write as before $q(x) = \langle q, v(x) \rangle = \langle q', v(x) \rangle$ ($x \in K$) and remark that:

$$\begin{aligned} \int_K qd\mu &= \int_K \langle q', v(x) \rangle d\mu(x) = \\ &= \sum_{j=1}^N c_j \langle q', v(x_j) \rangle = \sum_{j=1}^N c_j \langle q, v(x_j) \rangle = \\ &= \sum_{j=1}^N c_j q(x_j). \end{aligned}$$

Thus identity (2) holds for all polynomials $q \in P_d(\mathbf{R}^n)$. Finally, the quadrature relation (1) in the statement is obtained by neglecting in (2) the coefficients c_j which are zero. This completes the proof of Theorem 1.

Proof of Theorem 2. Let μ be a positive measure on \mathbf{R}^n which admits all its moments, up to order $2d$. Let r be a positive integer. We denote by μ_r the measure μ restricted to the closed ball centered at 0, of radius r .

Fix the positive integer r . According to Theorem 1, there is a system of N points ($N = N_{2d}(n)$), $x_j(r)$ ($\|x_j(r)\| \leq r$), and non-negative coefficients $c_j(r)$ (some of them possibly equal to zero) such that:

$$(3) \quad \int_{\|x\| \leq r} p d\mu = \sum_{j=1}^N c_j(r) p(x_j(r)) \quad (p \in P_{2d}(\mathbf{R}^n)).$$

Let $x_j(r) = (x_{j1}(r), \dots, x_{jn}(r))$ be the coordinates of the vectors appearing in formula (3). We fix an order in the above set of double indices $(j, i), 1 \leq j \leq N, 1 \leq i \leq n$, and pass successively, following this order, to subsequences in r , such that finally we have the following dichotomy for each pair (j, i) : either $\lim_{r \rightarrow \infty} x_{ji}(r)$ exists or $\lim_{r \rightarrow \infty} |x_{ji}(r)| = \infty$. In the preceding limits and throughout the rest of this proof it is implicitly assumed that r runs only on a subset of the positive integers.

Let C be a positive constant such that:

$$\int_{\mathbf{R}^n} |x^\alpha| d\mu(x) \leq C \quad (|\alpha| \leq 2d).$$

As a consequence of identity (3) and the fact that all coefficients $c_j(r)$ are non-negative we obtain:

$$c_j(r)(x_{ji}(r))^{2d} \leq C \quad (1 \leq j \leq N, 1 \leq i \leq n).$$

Suppose that for a pair of indices (j, i) we have $\lim_{r \rightarrow \infty} |x_{ji}(r)| = \infty$. Then

$$(4) \quad \lim_{r \rightarrow \infty} c_j(r)(x_{ji}(r))^k = 0 \quad (0 \leq k \leq 2d - 1).$$

On the other hand, identity (3) implies, for all r , that:

$$c_1(r) + \dots + c_N(r) \leq C.$$

Therefore, by passing to another subsequence in r we can assume that the limits

$$\lim_{r \rightarrow \infty} c_j(r) = c_j$$

exist for all $1 \leq j \leq N$.

Let $j, 1 \leq j \leq N$, be an index with the property that there exists $i, 1 \leq i \leq n$, such that $\lim_{r \rightarrow \infty} |x_{ji}(r)| = \infty$. Then relation (4) shows that $c_j = \lim_{r \rightarrow \infty} c_j(r) = 0$. Thus relation (4) holds for all indices i , regardless of whether the limit $\lim_{r \rightarrow \infty} |x_{ji}(r)|$ is finite or not. Consequently

$$\lim_{r \rightarrow \infty} [c_j(r) \max_{1 \leq i \leq n} |x_{ji}(r)|^{2d-1}] = 0.$$

For every multi-index $\alpha \in \mathbf{N}^n, |\alpha| \leq 2d - 1$, we have, for large values of r :

$$|c_j(r)x_j(r)^\alpha| \leq c_j(r) \max_{1 \leq i \leq n} |x_{ji}(r)|^{2d-1},$$

whence

$$\lim_{r \rightarrow \infty} c_j x_j(r)^\alpha = 0 \quad (|\alpha| \leq 2d - 1).$$

In other words, for the above fixed index j and for every polynomial $p \in P_{2d-1}(\mathbf{R}^n)$ we have proved that

$$\lim_{r \rightarrow \infty} c_j(r) p(x_{ji}(r)) = 0.$$

In the other case, when $x_j = \lim_{r \rightarrow \infty} x_j(r)$ exists, we have:

$$\lim_{r \rightarrow \infty} c_j(r) p(x_j(r)) = c_j p(x_j) \quad (p \in P_{2d-1}(\mathbf{R}^n)).$$

Let us denote by J the set of those indices $j, 1 \leq j \leq N$, in this latter category. Notice that for all $j \in J$ the limit point x_j is in the closed support of the measure μ .

In conclusion, the desired quadrature identity for a polynomial p of total degree less than or equal to $2d - 1$ is obtained as follows:

$$\int_{\mathbf{R}^n} p d\mu = \lim_{r \rightarrow \infty} \int_{|x| \leq r} p(x) d\mu(x) = \lim_{r \rightarrow \infty} \sum_{j=1}^N c_j(r) p(x_j(r)) = \sum_{j \in J} c_j p(x_j).$$

This finishes the proof of Theorem 2.

Corollary 1. *Let d be a positive integer and let μ be a positive measure supported by a proper closed convex cone of \mathbf{R}^n generated by n linearly independent vectors. Assume that the measure μ admits moments up to degree d . Then there are N points ($N \leq N_d(n)$), $x_j \in \text{supp}(\mu)$, and positive real numbers c_j ($1 \leq j \leq N$), such that the quadrature identity (1) holds for all $p \in P_{d-1}(\mathbf{R}^n)$.*

Proof. Since the statements of Theorem 2 and Corollary 1 are invariant under linear changes of coordinates, we can assume that the support of the measure μ is contained in the first octant $(\mathbf{R}_+)^n$. Hence, with the notation in the proof of Theorem 2, from the majorization

$$\int_{\mathbf{R}^n} |x^\alpha| d\mu(x) \leq C \quad (|\alpha| \leq d),$$

we still have

$$c_j(r)(x_{ji}(r))^d \leq C \quad (1 \leq j \leq N, 1 \leq i \leq n).$$

Then we repeat the rest of the proof of Theorem 2.

3. GAUSSIAN QUADRATURES

A central theme in the theory of quadrature identities is to construct and describe quadrature formulae with a minimal number of nodes. These formulae are called Gaussian quadratures by following the classical theory in one variable. A simple observation shows that, for a compactly supported positive measure in \mathbf{R}^n , the number of nodes N of a degree d quadrature formula with positive coefficients satisfies $N \geq N_{[d/2]}(n)$. By a widely accepted convention and terminology, the case $N = N_{[d/2]}$ corresponds to Gaussian quadratures; see for details [1], [7] and [8] and the references there.

According to Corollary 1, we obtain the following result.

Proposition 1. *Let n and d be positive integers and let μ be a measure supported by a proper closed convex cone in \mathbf{R}^n generated by n linearly independent vectors. Let $K_j, j \geq 1$, be an increasing compact exhaustion of \mathbf{R}^n .*

Suppose that the measure μ admits all moments of degree $2d + 1$ and that for each $j \geq 1$ the restriction $\mu|_{K_j}$ admits a Gaussian quadrature formula of degree $2d + 1$.

Then the measure μ admits a Gaussian quadrature formula of degree $2d$.

To prove Proposition 1 we remark that $[(2d + 1)/2] = [(2d)/2]$ and we apply Corollary 1.

Starting with Gaussian quadratures of degree $2d$ for the restricted measures $\mu|K_j$, the same technique yields for the full measure μ a quadrature formula of degree $2d - 1$ with $N_d(n)$ nodes instead of the required minimum of $N_{d-1}(n)$ nodes.

For the rather intricate structure of the Gaussian quadrature formulae and the present status of their theory the reader can profitably consult references [1], [4], [7] and [8].

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