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LIFTING OF GENERATING SUBGROUPS

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Abstract. Let $\varphi\colon G \to H$ be an epimorphism of finite groups. Suppose that G is generated by its subgroups $G_1,\dots,G_n$ and that H is generated by its subgroups $H_1,\dots,H_n$. Furthermore, suppose that $\varphi(G_i)$ and H_i are conjugate, $i=1,\dots,n$. We prove that there exist $g_1,\dots,g_n\in G$ such that $G_1^{g_1},\dots,G_n^{g_n}$ generate G and $\varphi(G_i^{g_i})=H_i$, $i=1,\dots,n$.

In this note we prove the following lifting principle for generating subgroups:

Main Theorem. *Let $\varphi\colon G\to H$ be an epimorphism of finite groups and let $G_1,\dots,G_n$ and $H_1,\dots,H_n$ be subgroups of G and H, respectively. Suppose that $G=\langle G_1,\dots,G_n\rangle$ and $H=\langle H_1,\dots,H_n\rangle$. Moreover, suppose that φ maps some conjugate of G_i onto H_i, $i=1,\dots,n$. Then there exist conjugates $G_1^{g_1},\dots,G_n^{g_n}$ of $G_1,\dots,G_n$ which are mapped by φ onto $H_1,\dots,H_n$, respectively, and which generate G.*

This theorem and its proof are motivated by the following result:

Theorem (Gaschütz [4]). *Let $\varphi\colon G\to H$ be an epimorphism of finite groups and let $\{h_1,\dots,h_n\}$ be a set of generators for H. Suppose that G can be generated by n elements. Then there exists a set $\{g_1,\dots,g_n\}$ of generators for G such that $\varphi(g_i)=h_i$, $i=1,\dots,n$.*

Gaschütz proved this fact by means of a sophisticated counting argument. His proof has later been simplified by Roquette [2, Lemma 15.30]. This theorem had an important and quite surprising role in the model theory of fields [3]. See also [1] for a generalization of Gaschütz' theorem, which however does not cover our main theorem.

Proof of the Main Theorem. Let J be the collection of all subgroups F of G such that $\varphi(F)=H$. For $F\in J$, $h\in H$ and a left coset $gF\in G/F$ (where $g\in G$) let

$$A_i(F,h,gF)=gF\cap\varphi^{-1}(N_H(\varphi(G_i))h),\quad i=1,\dots,n,$$

where N_H denotes the normalizer of a subgroup of H. Thus $g'\in A_i(F,h,gF)$ if and only if g and g' belong to the same left coset in G/F and $\varphi(G_i^{g'})=\varphi(G_i)^h$. We show that the size of the finite set $A_i(F,h,gF)$ does not depend on the choice of h. Indeed, for any other $h'\in H$ we can take $f,f'\in F$ with $\varphi(f)=h$ and $\varphi(f')=h'$. Then $A_i(F,h',gF)=A_i(F,h,gF)f^{-1}f'$.

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For $F\in J$, for $h_1,\dots,h_n\in H$, and for left cosets $g_1F,\dots,g_nF\in G/F$ (where $g_1,\dots,g_n\in G$) set:

$$\begin{aligned}B(F,h_1,\dots,h_n,&g_1F,\dots,g_nF)\\&=\Big\{(g'_1,\dots,g'_n)\in\prod_{i=1}^n A_i(F,h_i,g_iF)\ \Big|\ F=\langle G_1^{g'_1},\dots,G_n^{g'_n}\rangle\Big\}.\end{aligned}$$

Claim. Let $F\in J$, let $h_1,\dots,h_n\in H$ satisfy $H=\langle\varphi(G_1)^{h_1},\dots,\varphi(G_n)^{h_n}\rangle$, and let $g_i\in G$ satisfy $G_i^{g_i}\le F$, $i=1,\dots,n$. Then:

$$\begin{aligned}\prod_{i=1}^n A_i(F,h_i,g_iF)=\dot{\bigcup}\Big\{B(F^*,h_1,\dots,h_n,&g'_1F^*,\dots,g'_nF^*)\ \Big|\ F^*\in J,\\&F^*\le F,\ g'_iF=g_iF,\ i=1,\dots,n\Big\}.\end{aligned}$$

Indeed, it is straightforward to check that the right-hand side is contained in the left-hand side. Conversely, suppose that $(g'_1,\dots,g'_n)\in\prod_{i=1}^n A(F,h_i,g_iF)$ and set $F^*=\langle G_1^{g'_1},\dots,G_n^{g'_n}\rangle$. Clearly, $(g'_1,\dots,g'_n)\in B(F^*,h_1,\dots,h_n,g'_1F^*,\dots,g'_nF^*)$. In addition, $g'_iF=g_iF$ and $G_i^{g_i}\le F$, so one has $G_i^{g'_i}\le F$ for each $1\le i\le n$, whence $F^*\le F$. Also $g'_i\in A_i(F,h_i,g_iF)$ implies that $\varphi(G_i^{g'_i})=\varphi(G_i)^{h_i}$, and we obtain:

$$\varphi(F^*)=\langle\varphi(G_1^{g'_1}),\dots,\varphi(G_n^{g'_n})\rangle=\langle\varphi(G_1)^{h_1},\dots,\varphi(G_n)^{h_n}\rangle=H.$$

Therefore $F^*\in J$. Conclude that $(g'_1,\dots,g'_n)$ belongs to the right-hand side.

To see that this union is disjoint we need to prove that an n-tuple

$$(g''_1,\dots,g''_n)\in B(F^*,h_1,\dots,h_n,g'_1F^*,\dots,g'_nF^*)$$

already determines F^* and $g'_1F^*,\dots,g'_nF^*$. Now $F^*=\langle G_1^{g''_1},\dots,G_n^{g''_n}\rangle$, and as $g''_i\in A_i(F^*,h_i,g'_iF^*)$ we have $g'_iF^*=g''_iF^*$, $i=1,\dots,n$. This proves the claim.

By the assumption of the theorem there exist $h_1,\dots,h_n\in H$ with $\varphi(G_i)^{h_i}=H_i$, $i=1,\dots,n$. Therefore

$$H=\langle\varphi(G_1)^{h_1},\dots,\varphi(G_n)^{h_n}\rangle$$

as well as

$$H=\varphi(G)=\langle\varphi(G_1),\dots,\varphi(G_n)\rangle.$$

By the first part of the proof, $|A_i(F,h_i,g_iF)|=|A_i(F,1,g_iF)|$ for every $F\in J$ and cosets $g_iF\in G/F$, $i=1,\dots,n$. Assuming in addition that $G_i^{g_i}\le F$ for each i, the claim implies that

$$\begin{aligned}&|B(F,h_1,\dots,h_n,g_1F,\dots,g_nF)|+\sum\Big|B(F^*,h_1,\dots,h_n,g'_1F^*,\dots,g'_nF^*)\Big|\\=&\prod_{i=1}^n\big|A_i(F,h_i,g_iF)\big|=\prod_{i=1}^n\big|A_i(F,1,g_iF)\big|\\=&|B(F,1,\dots,1,g_1F,\dots,g_nF)|+\sum\Big|B(F^*,1,\dots,1,g'_1F^*,\dots,g'_nF^*)\Big|,\end{aligned}$$

where the sums are taken over all *proper* subgroups F^* of F which belong to J and over all left cosets $g'_1F^*,\dots,g'_nF^*$ in G/F^* satisfying $g'_iF=g_iF$, $i=1,\dots,n$.

Note that when $g'_i F = g_i F$ one has $G_i^{g'_i} \le F$ as well, $i = 1, \dots, n$. By induction on the size of $F \in J$ we now obtain that

$$|B(F, h_1, \dots, h_n, g_1 F, \dots, g_n F)| = |B(F, 1, \dots, 1, g_1 F, \dots, g_n F)|$$

for every $F \in J$ and $g_i F \in G/F$ which satisfy $G_i^{g_i} \le F$, $i = 1, \dots, n$. Since $(1, \dots, 1) \in B(G, 1, \dots, 1, G, \dots, G)$ also $B(G, h_1, \dots, h_n, G, \dots, G) \ne \emptyset$. But for every n-tuple $(g'_1, \dots, g'_n)$ in the latter set we have $G = \langle G_1^{g'_1}, \dots, G_n^{g'_n} \rangle$ as well as

$$\varphi(G_i^{g'_i}) = \varphi(G_i)^{h_i} = H_i,$$

$i = 1, \dots, n$. This gives conjugates of $G_1, \dots, G_n$ as required. □

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