

HYPERSURFACES IN \mathbb{R}^d AND THE VARIANCE OF EXIT TIMES FOR BROWNIAN MOTION

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ABSTRACT. Using the first exit time for Brownian motion from a smoothly bounded domain in Euclidean space, we define two natural functionals on the space of embedded, compact, oriented, unparametrized hypersurfaces in Euclidean space. We develop explicit formulas for the first variation of each of the functionals and characterize the critical points.

1. INTRODUCTION

In this note we study two natural variational problems on the space of embedded, compact, oriented, unparametrized hypersurfaces in \mathbb{R}^d which bound a connected open domain. We denote this space by $\mathcal{SH}(\mathbb{R}^d)$. Given $S \in \mathcal{SH}(\mathbb{R}^d)$, let $D = D(S)$ be the domain bounded by S and let $|D|$ be the volume of D . Let τ_S be the first exit time of Brownian motion from $D(S)$; that is, $\tau_S = \tau_S(\omega) = \inf\{t \geq 0 : W_t(\omega) \in S\}$ where W_t is d -dimensional Brownian motion.

We define a functional which is the average variance of τ_S , where the average is taken over all the starting points of Brownian motion. That is, $\mathcal{A} : \mathcal{SH}(\mathbb{R}^d) \rightarrow \mathbb{R}$ is defined by

$$\mathcal{A}(S) = \frac{1}{|D|} \int_D [E_x(\tau_S^2) - E_x(\tau_S)^2] dx,$$

where $E_x(\tau_S)$ is the expected value of the random variable τ_S , given that Brownian motion starts at x , and similarly for $E_x(\tau_S^2)$.

For v_0 an arbitrary constant, define $\mathcal{SH}_{v_0}(\mathbb{R}^d)$ by

$$\mathcal{SH}_{v_0}(\mathbb{R}^d) = \{S \in \mathcal{SH}(\mathbb{R}^d) : |D(S)| = v_0\}.$$

We characterize the critical points of \mathcal{A} in the following theorem.

Theorem 1.1. *Let $\mathcal{A} : \mathcal{SH}_{v_0}(\mathbb{R}^d) \rightarrow \mathbb{R}$ be defined as above. Then \mathcal{A} is a smooth functional and the critical points of \mathcal{A} are surfaces bounding domains of volume v_0*

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for which there is a solution of the boundary value problem

$$(1.1) \quad \Delta u + 2 = 0 \text{ and } \Delta w + u = 0 \text{ on } D(S),$$

$$(1.2) \quad u = 0 \text{ and } w = 0 \text{ on } S,$$

$$(1.3) \quad \frac{\partial u}{\partial \nu} \frac{\partial w}{\partial \nu} = k \text{ on } S,$$

where $\partial/\partial\nu$ is the normal derivative along S and k is a constant. Moreover, S is a critical point of \mathcal{A} if and only if S is a sphere, and the functional is maximized at spheres.

We prove Theorem 1.1 by obtaining an explicit formula for the first variation of the functional \mathcal{A} (see Proposition 3.1). Our strategy is similar to the approach used in [KM] for the average expectation functional, $\mathcal{B}(S) = \frac{1}{|D|} \int_D E_x(\tau_S) dx$. We use Itô's formula to express \mathcal{A} in terms of a solution of a Poisson problem on $D(S)$. We then use Hadamard's classic approach of domain variation to obtain formulas for the variation of the functional \mathcal{A} in terms of the Green's function for the domain $D(S)$. Finally, we use symmetric rearrangement to prove that spheres define global maxima for the functional \mathcal{A} .

Let $x_0 \in \mathbb{R}^d$ and consider the subspaces $\mathcal{SH}(\mathbb{R}^d, x_0) = \{S \in \mathcal{SH}(\mathbb{R}^d) : x_0 \in D(S)\}$ and $\mathcal{SH}_{v_0}(\mathbb{R}^d, x_0) = \mathcal{SH}(\mathbb{R}^d, x_0) \cap \mathcal{SH}_{v_0}(\mathbb{R}^d)$. Define a functional $\mathcal{V} : \mathcal{SH}(\mathbb{R}^d, x_0) \rightarrow \mathbb{R}$, the variance of the random variable τ_S , by

$$(1.4) \quad \mathcal{V}(S) = E_{x_0}(\tau_S^2) - E_{x_0}(\tau_S)^2.$$

Using techniques similar to those used to prove Theorem 1.1, it is straightforward to obtain a variational formula for \mathcal{V} , from which it follows that spheres are critical points for the functional \mathcal{V} . The converse statement, however, depends upon commutation properties of several classical operators.

Free boundary problems similar to (1.1)–(1.3) arise in a number of problems, in which one characterizes the extrema of a functional defined in terms of exit times for a Brownian particle ([KM], [FM]). Closely related free boundary problems were studied extensively by Serrin [S] and inspired a number of related semilinear results ([GNN]).

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2. BASIC RESULTS AND DEFINITIONS

Let (Ω, \mathcal{B}) be a measurable space and $\{P_x\}_{x \in \mathbb{R}^d}$ a family of probability measures on (Ω, \mathcal{B}) . Let $\{W_t\}_{t \geq 0}$ denote d -dimensional Brownian motion for which $P_x\{W_0 = x\} = 1$, for $x \in \mathbb{R}^d$. For each $x \in \mathbb{R}^d$, we will denote the expected value of a random variable Y under the probability measure P_x by $E_x(Y)$.

Let $\mathcal{SH}(\mathbb{R}^d)$ be as in the introduction, and note that $\mathcal{SH}(\mathbb{R}^d)$ is a Fréchet manifold. Given a point $S \in \mathcal{SH}(\mathbb{R}^d)$, we can identify the tangent space at S with the space of smooth vectorfields normal to S . The space of smooth vectorfields normal to S can be naturally identified with the space of smooth functions on S , and we obtain an identification: $T_S \mathcal{SH}(\mathbb{R}^d) \simeq C^\infty(S)$. By definition, it is clear that $\mathcal{SH}(\mathbb{R}^d, x_0)$ is an open set in $\mathcal{SH}(\mathbb{R}^d)$. Moreover, it is a Fréchet manifold with tangent space at S given by $C^\infty(S)$.

Given $S \in \mathcal{SH}(\mathbb{R}^d)$ and $f \in C^\infty(S)$, a normal variation of S in the direction of f is a one-parameter family $\{S^\epsilon\} \subset \mathcal{SH}(\mathbb{R}^d)$, $S^\epsilon = \{y \in \mathbb{R}^d : y = \sigma + \epsilon f(\sigma)\nu(\sigma), \sigma \in S\}$, where $\nu(\sigma)$ is the outward-pointing unit normal vector to S at σ and, for ϵ

sufficiently small, the map $F : S \rightarrow S^\epsilon$ defined by $F(\sigma) = y$ is a diffeomorphism. Normal variations allow us to introduce coordinates in a neighborhood of a point $S \in \mathcal{SH}(\mathbb{R}^d)$. The coordinates near S are given by the space $C^\infty(S)$, and it is in these coordinates that all the computations are carried out.

Given $S \in \mathcal{SH}(\mathbb{R}^d)$, the volume of $D(S)$ defines a smooth functional $|D| : \mathcal{SH}(\mathbb{R}^d) \rightarrow \mathbb{R}$. We say that a normal variation of S in direction f fixes volume if the variation of $|D|$ in the direction of f (denoted $(\delta|D|)f$) vanishes. A simple computation establishes that a normal variation in direction f fixes volume if and only if $\int_S f \, d\sigma = 0$. We will henceforth denote the space of smooth mean-zero functions on S by $\mathcal{M}(S) = \{f \in C^\infty(S) : \int_S f \, d\sigma = 0\}$. Note that the space $\mathcal{SH}_{v_0}(\mathbb{R}^d)$ is a Fréchet manifold with tangent space $T_S \mathcal{SH}_{v_0}(\mathbb{R}^d)$ at $S \in \mathcal{SH}_{v_0}(\mathbb{R}^d)$ identified with $\mathcal{M}(S)$. The space $\mathcal{SH}_{v_0}(\mathbb{R}^d, x_0)$ is an open set in $\mathcal{SH}_{v_0}(\mathbb{R}^d)$. It is also a Fréchet manifold with tangent space at S identified with $\mathcal{M}(S)$.

Given an element $S \in \mathcal{SH}(\mathbb{R}^d)$, there is a useful relationship between the solution of a certain Poisson problem on the domain $D(S)$ and the expected value of the first exit time of Brownian motion from $D(S)$ starting at $x \in D(S)$. Indeed, suppose u solves the problem

$$(2.1) \quad \begin{aligned} \Delta u + 2 &= 0 \text{ on } D(S), \\ u &= 0 \text{ on } S. \end{aligned}$$

Using Itô's formula we have

$$u(W_{\tau_S}) - u(W_0) = \int_0^{\tau_S} \nabla u(W_s) \cdot dW_s + \frac{1}{2} \int_0^{\tau_S} \Delta u(W_s) \, ds$$

where the first random variable, $\int_0^{\tau_S} \nabla u(W_s) \cdot dW_s$, is a stochastic integral taken with respect to Brownian motion. From this it follows that

$$\tau_S = u(W_0) + \int_0^{\tau_S} \nabla u(W_s) \cdot dW_s.$$

Let $X_t = \int_0^t \nabla u(W_s) \cdot dW_s$, and note that X_t is a mean-zero martingale. The above line can be rewritten as $\tau_S = u(x) + X_{\tau_S}$. If we square both sides of this equality and take expectations, we obtain

$$(2.2) \quad E_x(\tau_S^2) - [E_x(\tau_S)]^2 = E_x(X_{\tau_S}^2).$$

The quantity on the right of (2.2) can also be related to the solution of a Poisson problem:

$$\begin{aligned} E_x(X_{\tau_S}^2) &= E_x \left[\left(\int_0^{\tau_S} \nabla u(W_s) \cdot dW_s \right)^2 \right] \\ &= E_x \left[\int_0^{\tau_S} |\nabla u(W_s)|^2 \, ds \right] \\ &= -\alpha(x) \end{aligned}$$

where u solves (2.1) and α satisfies

$$(2.3) \quad \begin{aligned} \Delta \alpha &= 2|\nabla u|^2 \text{ on } D(S), \\ \alpha &= 0 \text{ on } S. \end{aligned}$$

In particular, we can express the functional $\mathcal{A} : SH(\mathbb{R}^d) \rightarrow \mathbb{R}$ as $\mathcal{A}(S) = -\frac{1}{|D|} \int_D \alpha(x) dx$ and the functional $\mathcal{V} : SH(\mathbb{R}^d, x_0) \rightarrow \mathbb{R}$ as $\mathcal{V}(S) = -\alpha(x_0)$ where α solves (2.3). This establishes that the functionals \mathcal{A} and \mathcal{V} are smooth.

3. FIRST VARIATION OF \mathcal{A} AND \mathcal{V}

In order to compute the variation of \mathcal{A} and characterize the critical points, we first establish an equivalent expression for \mathcal{A} .

Lemma 3.1. *Let $S \in SH(\mathbb{R}^d)$ and suppose that \mathcal{A} is defined as above. Then*

$$\mathcal{A}(S) = \frac{1}{|D|} \int_D u^2(x) dx,$$

where u solves (2.1).

Proof. Recall that $\mathcal{A}(S) = -\frac{1}{|D|} \int_D \alpha(x) dx$, where α solves (2.3). We can represent α using the Green’s function for the Laplacian on $D(S)$ by

$$\alpha(x) = 2 \int_D |\nabla u(y)|^2 G(x, y) dy$$

(note that $G(x, y) \leq 0$). Hence,

$$\begin{aligned} \frac{1}{|D|} \int_D \alpha(x) dx &= -\frac{1}{|D|} \int_D |\nabla u(y)|^2 u(y) dy \\ (3.1) \qquad \qquad \qquad &= -\frac{1}{2} \frac{1}{|D|} \int_D \langle \nabla(u^2), \nabla u \rangle dy. \end{aligned}$$

By the Divergence Theorem and the fact that $u = 0$ on S ,

$$-\frac{1}{2} \frac{1}{|D|} \int_D \langle \nabla(u^2), \nabla u \rangle dy = \frac{1}{2} \frac{1}{|D|} \int_D u^2 \Delta u dy.$$

Hence, using that $\Delta u = -2$ on D ,

$$\frac{1}{|D|} \int_D \alpha(x) dx = -\frac{1}{|D|} \int_D u^2(x) dx,$$

which completes the proof.

Proposition 3.1. *Let $S \in SH_{v_0}(\mathbb{R}^d)$ and suppose $\delta\mathcal{A} : T_S SH_{v_0}(\mathbb{R}^d) \rightarrow \mathbb{R}$ denotes the variation of \mathcal{A} at S . Then*

$$(\delta\mathcal{A})f = \frac{2}{|D|} \int_S f \frac{\partial u}{\partial \nu} \frac{\partial w}{\partial \nu} d\sigma,$$

where u solves (2.1) and w solves the problem

$$(3.2) \qquad \begin{aligned} \Delta w + u &= 0 \text{ on } D(S), \\ w &= 0 \text{ on } S. \end{aligned}$$

Proof. Let $f \in C^\infty(S)$ be a tangent vector and let S^ϵ be a normal variation in the direction of f . By Lemma 3.1 it suffices to compute

$$(3.3) \qquad \begin{aligned} \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{D^\epsilon} u_\epsilon^2 dx &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left[\int_{D^\epsilon} u_\epsilon^2 dx - \int_D u^2 dx \right] \\ &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left[\int_{D^\epsilon \cap D} (u_\epsilon^2 - u^2) dx + \int_{D^\epsilon \setminus D} u_\epsilon^2 dx - \int_{D \setminus D^\epsilon} u^2 dx \right]. \end{aligned}$$

Since u_ϵ vanishes on S^ϵ and u vanishes on S , $\int_{D^\epsilon \setminus D} u_\epsilon^2 dx - \int_{D \setminus D^\epsilon} u^2 dx = o(\epsilon)$. Hence,

$$\begin{aligned} \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{D^\epsilon} u_\epsilon^2 dx &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{D^\epsilon \cap D} (u_\epsilon^2 - u^2) dx \\ &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{D^\epsilon \cap D} (u_\epsilon - u)(u_\epsilon + u) dx \\ &= - \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{D^\epsilon \cap D} (u_\epsilon - u) \Delta(w_\epsilon + w) dx. \end{aligned}$$

Since $u_\epsilon - u$ is harmonic on $D^\epsilon \cap D$ we can use Green's theorem to write

$$\begin{aligned} \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{D^\epsilon} u_\epsilon^2 dx &= - \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{\partial(D^\epsilon \cap D)} \left((u_\epsilon - u) \frac{\partial(w_\epsilon + w)}{\partial n} - (w_\epsilon + w) \frac{\partial(u_\epsilon - u)}{\partial n} \right) dz \end{aligned}$$

where n is the outward-pointing unit normal to $\partial(D^\epsilon \cap D)$. Since w_ϵ vanishes on S^ϵ and w vanishes on S , and, since $\frac{\partial(u_\epsilon - u)}{\partial n} = O(\epsilon)$ uniformly on $\partial(D^\epsilon \cap D)$, we have

$$\frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{D^\epsilon} u_\epsilon^2 dx = - \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{\partial(D^\epsilon \cap D)} (u_\epsilon - u) \frac{\partial(w_\epsilon + w)}{\partial n} dz.$$

We can partition $\partial(D^\epsilon \cap D)$ as $\partial(D^\epsilon \cap D) = S_+ \cup S_-^\epsilon$ where

$$\begin{aligned} S_+ &= \{\sigma \in S : f(\sigma) \geq 0\}, \\ S_- &= \{\sigma \in S : f(\sigma) < 0\}, \\ S_-^\epsilon &= \{y = \sigma + \epsilon f(\sigma) \nu(\sigma) \in S^\epsilon : \sigma \in S_-\}. \end{aligned}$$

Using the boundary conditions,

$$\begin{aligned} (3.4) \quad \int_{\partial(D^\epsilon \cap D)} (u_\epsilon - u) \frac{\partial(w_\epsilon + w)}{\partial n} dz &= \int_{S_+} u_\epsilon \frac{\partial(w_\epsilon + w)}{\partial \nu} d\sigma - \int_{S_-^\epsilon} u \frac{\partial(w_\epsilon + w)}{\partial \nu_\epsilon} dz \\ &= \int_{S_+} u_\epsilon \frac{\partial(w_\epsilon + w)}{\partial \nu} d\sigma - \int_{S_-} u(y) \frac{\partial(w_\epsilon + w)}{\partial \nu_\epsilon}(y) J_\epsilon d\sigma \end{aligned}$$

where J_ϵ is the Jacobian of the map $S \ni \sigma \rightarrow y = \sigma + \epsilon f(\sigma) \nu(\sigma) \in S^\epsilon$. Expanding $u(\sigma + \epsilon f(\sigma) \nu(\sigma))$ about σ , $u(y) = u(\sigma) + \epsilon f(\sigma) \frac{\partial u}{\partial \nu}(\sigma) + o(\epsilon)$. Noting that $J_\epsilon = 1 + O(\epsilon)$ and that $\frac{\partial w_\epsilon}{\partial n} = \frac{\partial w}{\partial n} + O(\epsilon)$, where the $O(\epsilon)$ term is uniformly bounded on $\partial(D^\epsilon \cap D)$, we substitute in the second term of (3.4) to obtain

$$(3.5) \quad \int_{S_-} u(y) \frac{\partial(w_\epsilon + w)}{\partial \nu_\epsilon}(y) J_\epsilon d\sigma = 2\epsilon \int_{S_-} f(\sigma) \frac{\partial u}{\partial \nu}(\sigma) \frac{\partial w}{\partial \nu}(\sigma) d\sigma + o(\epsilon).$$

A similar computation for the first term in (3.4) gives

$$(3.6) \quad \int_{S_+} u_\epsilon \frac{\partial(w_\epsilon + w)}{\partial \nu} d\sigma = -2\epsilon \int_{S_+} f(\sigma) \frac{\partial u}{\partial \nu}(\sigma) \frac{\partial w}{\partial \nu}(\sigma) d\sigma + o(\epsilon).$$

Combining (3.4), (3.5) and (3.6),

$$(3.7) \quad \int_{\partial(D^\epsilon \cap D)} (u_\epsilon - u) \frac{\partial(w_\epsilon + w)}{\partial n} dz = -2\epsilon \int_S f(\sigma) \frac{\partial u}{\partial \nu}(\sigma) \frac{\partial w}{\partial \nu}(\sigma) d\sigma + o(\epsilon).$$

From (3.3), (3.7) and Lemma 3.1, the variation of $\mathcal{A} : \mathcal{S}H_{v_0}(\mathbb{R}^d) \rightarrow \mathbb{R}$ is therefore

$$(\delta\mathcal{A})f = \frac{2}{|D|} \int_S f \frac{\partial u}{\partial \nu} \frac{\partial w}{\partial \nu} d\sigma,$$

where u solves (2.1) and w solves (3.2).

Proof of Theorem 1.1. From Proposition 3.1 and the characterization of the tangent space of $\mathcal{S}H_{v_0}(\mathbb{R}^d)$ as the space of smooth mean-zero functions on S , we see that S is a critical point if and only if $\frac{\partial u}{\partial \nu} \frac{\partial w}{\partial \nu}$ is a constant. Clearly, if S is a sphere, then u and w are radial and $\frac{\partial u}{\partial \nu} \frac{\partial w}{\partial \nu}$ is a constant. In particular, spheres are critical points for the functional \mathcal{A} . Conversely, if the overdetermined boundary value problem (1.1)–(1.3) admits a solution, it follows that the domain is a sphere [FM].

To prove that spheres bounding balls of volume v_0 define global maxima for the functional \mathcal{A} , we use the notion of symmetric rearrangement. For the benefit of the reader we recall the definition and the basic facts we will need concerning rearrangements. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^+$ and denote by $\chi_{\{f>\gamma\}}(x)$ the characteristic function of the set $\{x : f(x) > \gamma\}$. Define the radius function associated to f , $R : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, by

$$\omega_d R(\gamma)^d = \int \chi_{\{f>\gamma\}} d\lambda_d$$

where ω_d is the volume of the unit ball in \mathbb{R}^d and λ_d denotes d -dimensional Lebesgue measure. Define the symmetric rearrangement of f , denoted f^* , by

$$f^*(x) = \int_{\gamma>0} \chi_{[0,R(\gamma)]}(|x|) d\gamma.$$

In order to prove Theorem 1.1, we shall use the fact that, for $S \in \mathcal{S}H_{v_0}(\mathbb{R}^d)$, $\mathcal{A}(S) = \frac{2}{|D|} \int w dx$, where w solves (3.2). This follows by Lemma 3.1 and Green's Theorem, since

$$\int_D u^2 dx = - \int_D u \Delta w dx = 2 \int_D w dx.$$

In addition, we will use the following theorem [B]:

Theorem 3.1 ([B]). *Let D be a smoothly bounded domain in \mathbb{R}^d with boundary of D denoted by S , and let v be the solution of*

$$\begin{aligned} \Delta v + f &= 0 \text{ on } D, \\ v &= 0 \text{ on } S, \end{aligned}$$

where f is a nonnegative Hölder continuous function on D . Let f^* be the symmetric rearrangement of f , defined on the sphere $S^* \in \mathcal{S}H_{v_0}(\mathbb{R}^d)$, and let \tilde{v} be the solution of the symmetrized problem

$$\begin{aligned} \Delta \tilde{v} + f^* &= 0 \text{ on } D^*, \\ \tilde{v} &= 0 \text{ on } S^*, \end{aligned}$$

where $D^* = D(S^*)$. If v^* denotes the symmetrization of v , then $\tilde{v} \geq v^*$.

We will also need the following lemma, which follows from the maximum principle.

Lemma 3.2. *Let D be a smoothly bounded domain in \mathbb{R}^d and suppose f_1 and f_2 are nonnegative Hölder continuous functions on D satisfying $f_1 \geq f_2$. Suppose v_1 and v_2 satisfy*

$$\begin{aligned} \Delta v_i + f_i &= 0 \text{ on } D, \\ v_i &= 0 \text{ on } S. \end{aligned}$$

Then $v_1 \geq v_2$ on D .

Let w solve (3.2) for $S \in \mathcal{S}H_{v_0}(\mathbb{R}^d)$, and let \tilde{w} be the solution of the symmetrized problem

$$\begin{aligned} \Delta \tilde{w} + u^* &= 0 \text{ on } D^*, \\ \tilde{w} &= 0 \text{ on } S^*. \end{aligned}$$

Recall that S^* is a sphere. From Theorem 3.1, we have $\tilde{w} \geq w^*$, and hence

$$(3.9) \quad \frac{1}{|D^*|} \int_{D^*} \tilde{w} \geq \frac{1}{|D^*|} \int_{D^*} w^* = \frac{1}{|D|} \int_D w,$$

where we have used that rearrangement preserves L^p norms. Let W be the solution of

$$\begin{aligned} \Delta W + U &= 0 \text{ on } D^*, \\ W &= 0 \text{ on } S^*, \end{aligned}$$

where U satisfies (2.1) on D^* . Recall that u solves (2.1) on D , and let \tilde{u} be the corresponding solution to the symmetrized problem as in Theorem 3.1. Now constant functions are invariant under symmetrization, and so \tilde{u} satisfies (2.1) and $U = \tilde{u}$ on D^* .

Thus, by Theorem 3.1, $\tilde{u}(x) \geq u^*(x)$ for all $x \in D^*$ and so $U \geq u^*$. Using Lemma 3.2, we have $W \geq \tilde{w}$. Integrating over D^* and using (3.9), we obtain

$$\mathcal{A}(S^*) = \frac{2}{|D^*|} \int_{D^*} W \geq \frac{2}{|D^*|} \int_{D^*} \tilde{w} \geq \frac{2}{|D|} \int_D w = \mathcal{A}(S).$$

Since S was arbitrary, this concludes the proof of Theorem 1.1.

Next, we examine the variation of the functional \mathcal{V} as defined in (1.4).

Proposition 3.2. *Let $S \in \mathcal{S}H_{v_0}(\mathbb{R}^d)$ and suppose $\delta\mathcal{V} : T_S\mathcal{S}H_{v_0}(\mathbb{R}^d) \rightarrow \mathbb{R}$ denotes the variation of \mathcal{V} at S . Then*

$$(3.10) \quad \begin{aligned} (\delta\mathcal{V})(f) &= 2 \int_S f(\sigma) \left[u(x_0) \frac{\partial u}{\partial \nu}(\sigma) \frac{\partial G}{\partial \nu}(x_0, \sigma) \right. \\ &\quad \left. - 2 \frac{\partial w}{\partial \nu}(\sigma) \frac{\partial G}{\partial \nu}(x_0, \sigma) + 2 \frac{\partial u}{\partial \nu}(\sigma) \frac{\partial H}{\partial \nu}(\sigma) \right] d\sigma. \end{aligned}$$

where $\frac{\partial}{\partial \nu}$ is the outward-pointing normal derivative, G is the Green's function for the Laplacian on $D = D(S)$, w solves (3.2), u solves (2.1) and H solves

$$(3.11) \quad \begin{aligned} \Delta H &= G(x_0, y) \text{ on } D, \\ H &= 0 \text{ on } S. \end{aligned}$$

Proof. Let $f \in C^\infty(S)$ be a tangent vector and let S^ϵ be a normal variation in the direction of f . To ease notation, assume that $f \leq 0$. The case of general f is handled as in the proof of Proposition 3.1. Recall that $\mathcal{V}(S) = -\alpha(x_0)$ where α is the solution of (2.1). Using the identity $2|\nabla u|^2 = 4u + \Delta u^2$, we obtain $\mathcal{V}(S) = 4w(x_0) - u^2(x_0)$, where u solves (2.1) and w solves (3.2). In [KM], we compute the variation of the functional $\mathcal{E}(S) = u(x_0)$ and we characterize the critical points as spheres, where the functional attains a maximum. Indeed, we show that the variation of the functional $\mathcal{E}(S) = u(x_0)$ is given by

$$(3.12) \quad (\delta\mathcal{E})(f) = - \int_S f(\sigma) \frac{\partial u}{\partial \nu}(\sigma) \frac{\partial G}{\partial \nu}(x_0, \sigma) \, d\sigma.$$

Moreover, recall that we can express $w(x_0)$ using the Green’s function for the Laplacian on D by

$$w(x_0) = - \int_D u(y)G(x_0, y)dy.$$

Then, letting D^ϵ denote $D(S^\epsilon)$ and G^ϵ denote the Green’s function for the Laplacian on D^ϵ , we have

$$\begin{aligned} & - \frac{d}{d\epsilon} \Big|_{\epsilon=0} w_\epsilon(x_0) \\ &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left[\int_{D^\epsilon} u_\epsilon(y)G_\epsilon(x_0, y) - u(y)G(x_0, y)dy - \int_{D \setminus D^\epsilon} u(y)G(x_0, y)dy \right] \\ &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left[\int_{D^\epsilon} (u_\epsilon(y) - u(y)) G_\epsilon(x_0, y) - u(y) (G(x_0, y) - G_\epsilon(x_0, y)) \, dy \right] \\ & - \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left[\int_{D \setminus D^\epsilon} u(y)G(x_0, y)dy \right]. \end{aligned}$$

Since $u(y)$ and $G(x_0, y)$ both vanish for $y \in S$, $\int_{D \setminus D^\epsilon} u(y)G(x_0, y)dy = o(\epsilon)$ and

$$(3.13) \quad \begin{aligned} - \frac{d}{d\epsilon} \Big|_{\epsilon=0} w_\epsilon(x_0) &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left[\int_{D^\epsilon} (u_\epsilon(y) - u(y)) G_\epsilon(x_0, y)dy \right] \\ &+ \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left[\int_{D^\epsilon} -u(y) (G(x_0, y) - G_\epsilon(x_0, y)) \, dy \right]. \end{aligned}$$

We first investigate the second term of (3.13). Observe that $G(x_0, y) - G_\epsilon(x_0, y)$ is harmonic on D^ϵ . Using Green’s Theorem we have

$$\begin{aligned} \int_{D^\epsilon} u(y) (G(x_0, y) - G_\epsilon(x_0, y)) \, dy &= \int_{S^\epsilon} w \frac{\partial(G(x_0, z) - G_\epsilon(x_0, z))}{\partial \nu_\epsilon} \, dz \\ &- \int_{S^\epsilon} (G(x_0, z) - G_\epsilon(x_0, z)) \frac{\partial w}{\partial \nu_\epsilon} \, dz, \end{aligned}$$

where dz is surface measure on S^ϵ . Since w vanishes on S and $\frac{\partial(G(x_0, z) - G_\epsilon(x_0, z))}{\partial \nu_\epsilon}$ is $O(\epsilon)$ uniformly on S_ϵ , we have

$$\int_{D^\epsilon} u(y) (G(x_0, y) - G_\epsilon(x_0, y)) \, dy = - \int_{S^\epsilon} (G(x_0, z) - G_\epsilon(x_0, z)) \frac{\partial w}{\partial \nu_\epsilon} \, dz + o(\epsilon).$$

Expanding near $\epsilon = 0$, $G(x_0, z) = G(x_0, \sigma) + \epsilon f(\sigma) \frac{\partial G(x_0, \sigma)}{\partial \nu} + o(\epsilon)$. Since $G_\epsilon(x_0, z) = 0$ for $z \in S^\epsilon$, we have

$$\int_{D^\epsilon} u(y) (G(x_0, y) - G_\epsilon(x_0, y)) dy = -\epsilon \int_S f(\sigma) \frac{\partial w}{\partial \nu} \frac{\partial G(x_0, \sigma)}{\partial \nu} d\sigma + o(\epsilon),$$

where we have used that the Jacobian of the transformation $S \ni \sigma \rightarrow z = \sigma + \epsilon f(\sigma)\nu(\sigma) \in S^\epsilon$ is $1 + O(\epsilon)$. In particular,

$$(3.14) \quad \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left[\int_{D^\epsilon} u(y) (G(x_0, y) - G_\epsilon(x_0, y)) dy \right] = - \int_S f(\sigma) \frac{\partial w}{\partial \nu} \frac{\partial G(x_0, \sigma)}{\partial \nu} d\sigma.$$

In a similar fashion, we investigate the first term of (3.13). Let H_ϵ be the solution of

$$\begin{aligned} \Delta H_\epsilon &= G_\epsilon(x_0, y) \text{ on } D^\epsilon, \\ H_\epsilon &= 0 \text{ on } S^\epsilon. \end{aligned}$$

Then, using that $u_\epsilon - u$ is harmonic on D^ϵ ,

$$\begin{aligned} \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{D^\epsilon} (u_\epsilon(y) - u(y)) G_\epsilon(x_0, y) dy &= \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{S^\epsilon} (u_\epsilon - u) \frac{\partial H_\epsilon}{\partial \nu_\epsilon} - H_\epsilon \frac{\partial (u_\epsilon - u)}{\partial \nu_\epsilon} dz \\ &= - \frac{d}{d\epsilon} \Big|_{\epsilon=0} \left[\int_{S^\epsilon} u \frac{\partial H_\epsilon}{\partial \nu_\epsilon} dz \right]. \end{aligned}$$

Expanding about $\epsilon = 0$, $u(z) = u(\sigma) + \epsilon f(\sigma) \frac{\partial u}{\partial \nu} + o(\epsilon)$. Noting that $\frac{\partial H_\epsilon}{\partial \nu_\epsilon} = \frac{\partial H}{\partial \nu} + O(\epsilon)$ and that the Jacobian of the map $S \rightarrow S^\epsilon$ is $1 + O(\epsilon)$ we obtain

$$(3.15) \quad \frac{d}{d\epsilon} \Big|_{\epsilon=0} \int_{D^\epsilon} (u_\epsilon(y) - u(y)) G_\epsilon(x_0, y) dy = - \int_S f(\sigma) \frac{\partial u}{\partial \nu} \frac{\partial H}{\partial \nu} d\sigma.$$

Combining (3.11), (3.12), (3.13), (3.14), and (3.15), we obtain (3.10).

Proposition 3.2 can be used to classify the critical points of the functional \mathcal{V} . In particular, note that if S is a sphere, $\frac{\partial w}{\partial \nu}$, $\frac{\partial G}{\partial \nu}(x_0, \cdot)$, $\frac{\partial u}{\partial \nu}$, and $\frac{\partial H}{\partial \nu}$ are all constant and hence $\delta\mathcal{V} : \mathcal{M}(S) \rightarrow \mathbb{R}$ is zero. We record this as a corollary.

Corollary 3.2. *Spheres are critical points for the functional \mathcal{V} .*

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