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A COUNTEREXAMPLE TO THE EXISTENCE OF PEAKING FUNCTIONS

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ABSTRACT. We construct a smoothly bounded pseudoconvex domain whose boundary contains no complex analytic variety such that some boundary point admits no holomorphic peak function.

1. Introduction and results

Let Ω be a domain in $\mathbb{C}^n$. Denote by $A(\Omega)$ the Banach algebra of continuous functions on $\overline{\Omega}$ which are holomorphic in Ω. We say that a boundary point p is a *peak point* for Ω if there is a function $f \in A(\Omega)$ such that: $f(p) = 1$ and $|f(z)| < 1$ on $\overline{\Omega} \setminus \{p\}$. The function f is called a *peaking function*.

A basic problem concerning peak points is to determine a necessary and sufficient condition for a boundary point of a pseudoconvex domain to be a peak point. There are several known sufficient conditions [1, 5, 6, 7, 12, 13, 17, 27]. For example, if $p \in b\Omega$ is strongly pseudoconvex point [20] or more generally if p is *h-extendible* [5, 27], then it is peak point. On the other hand, the maximum principle implies that if $p \in b\Omega$ is a peak point, then the boundary of Ω cannot contain any nontrivial complex analytic variety containing p as an interior point. However, it is still unknown what kind of non-existence of analytic structures is necessary. Sibony constructed an example in [22] for which some boundary point is a local peak point but not a global one. Nevertheless, the boundary of his example contains complex discs. In this note, we would like to provide an example whose boundary contains no complex analytic variety, but not every boundary point is a peak point, not even a local peak point. As a matter of fact, our example domain will be B-regular which means roughly that there is no potential theory in the boundary [23]. As far as smooth peak functions are concerned, there are already many counterexamples, see [8, 14, 18].

Recall that a domain Ω is *B-regular* [22, 23, 4] at $p \in b\Omega$ if there is a plurisubharmonic (**psh**) function u on Ω continuous on $\overline{\Omega}$ such that $u(p) = 0$ and $u < 0$ on $\overline{\Omega} \setminus \{p\}$. This is equivalent to saying that there is no nontrivial Jensen measure for p supported on $b\Omega$ (see [23]). Clearly, a peak point is necessarily B-regular. Moreover, the maximum principle excludes the existence of any analytic variety in

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the boundary around a B-regular point. B-regular domains are an important class of weakly pseudoconvex domains on which the $\overline{\partial}$-Neumann operator is well behaved (satisfies the compactness estimates) [4].

Our main result is

Theorem. *There are a smoothly bounded B-regular pseudoconvex domain in $\mathbb{C}^3$ and a point $p \in b\Omega$ such that p is not a peak point for $\Omega \cap U$ for any neighborhood U of p.*

2. The construction of the example

First of all, we fix some notations. Let K be a compact subset of $\mathbb{C}^n$. As usual, $C(K)$ is the Banach algebra of continuous functions on K. $\mathcal{O}(K)$ is the (uniform) closure in $C(K)$ of the functions holomorphic in some neighborhood of K. $PSH(K)$ is the closure in $C(K)$ of continuous psh functions in some neighborhood of K. $\mathcal{P}(K)$ is the closure in $C(K)$ of the set of polynomials in $\mathbb{C}^n$, and $\mathcal{R}(K)$ stands for the closure in $C(K)$ of the set of rational functions which have no poles in K.

The example desired will be made from the well-known “Swiss cheese” constructed by McKissick [16]. First we claim that

Lemma 1. *There are a polynomially convex compact subset K in $\mathbb{C}^2$ and $z_0 \in K$ satisfying*

- (i) *K is B-regular, i.e. every point is a peak point for $PSH(K)$;*
- (ii) *There is a nontrivial representing measure for z_0 for $\mathcal{O}(K)$ with support in an arbitrarily small neighborhood of z_0 in K.*

Here and below, “nontrivial” means “not a point mass”.

In order to construct such a compact set, we first choose a compact set X in $\mathbb{C}$ so that $\mathcal{R}(X)$ is *normal* but $\mathcal{R}(X) \neq C(X)$. Here “normal” means that for every pair of disjoint closed subsets E_0, E_1 of X, there is a function $r \in \mathcal{R}(X)$ such that $r \equiv i$ on E_i, $i = 0, 1$. Such a compact planar set was first found by McKissick [16] using a special “Swiss cheese” of the form $\overline{\Delta} \setminus \cup \Delta_j$, where each Δ_j is a small open disc inside the unit disc Δ. In particular, $X \subset \overline{\Delta}$. By a theorem of Rossi [20] (see also [25, 26]), $\mathcal{R}(X)$ is a doubly generated Banach algebra. More precisely, there are $r_1, r_2 \in \mathcal{R}(X)$, such that $\mathcal{R}(X) = \mathbb{C}[r_1, r_2]$. In fact, one can even choose $r_1(\zeta) = \zeta$.

Now define a map $\Phi : X \to \mathbb{C}^2$ by $\Phi(\zeta) = (r_1(\zeta), r_2(\zeta))$. Then Φ is a holomorphic embedding of X in $\mathbb{C}^2$. Denote its image by $K = \Phi(X)$. We will show that such K satisfies the conditions in Lemma 1. Observe that the compact set K is polynomially convex, i.e., for any $z \notin K$, there is a polynomial P such that $|P(z)| > \|P\|_{C(K)}$. This follows from the fact that the joint spectrum of a set of generators for a commutative Banach algebra with identity is always polynomially convex and the fact that the spectrum of $\mathcal{R}(X)$ is X itself (see, e.g., [25] for details).

We first verify that K is B-regular. Suppose not. Then in light of an equivalent condition for B-regularity given in [23], there are a point $z_0 \in K$ and a nontrivial Jensen measure μ for z_0 on K for $PSH(K)$. That is, the measure μ is a probability measure on K that satisfies

$$u(z_0) \leq \int_K u(z)\, d\mu, \quad \forall u \in PSH(K). \tag{1}$$

Since μ is not a point mass, there must be two disjoint closed subsets F_0, F_1 of K such that $\mu(F_i) > 0$, $i = 0, 1$. Set $E_i = \Phi^{-1}(F_i) \cap X$. Then E_0, E_1 are disjoint and compact. It follows from the normality of $\mathcal{R}(X)$ that there exists $r \in \mathcal{R}(X)$ such that $r \equiv i$ on E_i for $i = 0, 1$. On the other hand, since $\mathcal{R}(X)$ is generated as a Banach algebra by r_1, r_2, there exist polynomials g_n in $\mathbb{C}^2$ such that $r(\zeta) = \lim_{n\to\infty} g_n(r_1(\zeta), r_2(\zeta))$, uniformly on X. For any integers $m, n \geq 1$, the function $u_{mn} := \log(|g_n| + 1/m)$ is continuous and psh in $\mathbb{C}^2$. Applying (1) to u_{mn}, we obtain

$$\log(|g_n(z_0)| + \frac{1}{m}) \leq \int_K \log(|g_n(z)| + \frac{1}{m})\, d\mu.$$

If we set $\sigma = \mu \circ \Phi$, then σ is a probability measure on X such that $\sigma(E_i) = \mu(F_i) > 0$ for $i = 0, 1$. Let $\zeta_0 = \Phi^{-1}(z_0)$. A simple change of variables (see e.g. [11]) yields

$$\log(|g_n(\Phi(\zeta_0))| + \frac{1}{m}) \leq \int_X \log(|g_n(\Phi(\zeta))| + \frac{1}{m})\, d\sigma.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\log(|r(\zeta_0)| + \frac{1}{m}) \leq \int_X \log(|r(\zeta)| + \frac{1}{m})\, d\sigma.$$

Observe that the integrand in the above integral is bounded from above by the constant $\log(\|r\|_X + 1)$ and is non-increasing with respect to m. The monotone convergence theorem (see [11]) implies that

$$\log|r(\zeta_0)| \leq \int_X \log|r(\zeta)|\, d\sigma.$$

The right hand side of the above inequality is $-\infty$ due to the fact that $r(E_0) = \{0\}$ and $\sigma(E_0) > 0$. Thus we have $r(\zeta_0) = 0$. On the other hand, if we replace g_n by $1 - g_n$ in the above arguments, we will also have $1 - r(\zeta_0) = 0$. This is clearly impossible. Consequently, K must be B-regular.

In order to verify (ii) in Lemma 1, we recall that for a compact planar set Y, $\mathcal{R}(Y) = C(Y)$ if and only if almost all points (in the Lebesgue measure) of Y are peak points for $\mathcal{R}(Y)$ (see [25]). Since for our X, $\mathcal{R}(X) \neq C(X)$, it follows that there exists at least one point $\zeta_0 \in X$ that is not a peak point for $\mathcal{R}(X)$. Necessarily $\zeta_0 \in \Delta$. We may assume that $X \ni \zeta_0 = 0$ by a simple Möbius transform.

Let $X_t = \{\zeta \in X : |\zeta| \leq t\}$ for $t > 0$. Since $0 = \zeta_0$ is not a peak point for $\mathcal{R}(X)$, we claim that it is not a peak point for $\mathcal{R}(X_t)$ either, for any $t > 0$. Assume the contrary. Then there is a function $g \in \mathcal{R}(X_{t_0})$ for some $0 < t_0 < 1$ such that g peaks at 0. Choose t_1, t_2 such that $0 < t_2 < t_1 < t_0$. Set $E_1 = X_{t_2}$ and $E_0 = \overline{X \setminus X_{t_1}}$. The normality of the algebra $\mathcal{R}(X)$ yields a function $r \in \mathcal{R}(X)$ with $r(E_i) = \{i\}$ for $i = 0, 1$. Define a function h by setting $h = rg$ on X_{t_0} and $h = 0$ otherwise. It follows from the localization theorem for $\mathcal{R}(X)$ (see e.g. Corollary 10.3 in [10] or Theorem 26.1 in [25]) that $h \in \mathcal{R}(X)$. Since $h = g$ on X_{t_2}, h is also a local peak function for 0. Then by the local maximum modulus principle of Rossi [21], we know that 0 is also a peak point for $\mathcal{R}(X)$, which is a contradiction.

As a consequence of a theorem of Bishop [2] (see also [9]), there is a nontrivial representing measure σ_t for 0 on X_t for the algebra $\mathcal{R}(X_t)$, for any $t > 0$.

That is,

$$g(0) = \int_{X_t} g(\zeta)\, d\sigma, \quad \forall g \in \mathcal{R}(X_t). \tag{3}$$

Set $\mu_t = \sigma_t \circ \Phi^{-1}$. Then μ_t is a well defined nontrivial probability measure on $K_t = \Phi(X_t)$. For any $f \in \mathcal{P}(K_t)$, let $g = f \circ \Phi \in \mathcal{R}(X_t)$. A simple application of change of coordinates for measurable mapping [11] to (3) yields

$$f(\Phi(0)) = \int_{X_t} f(\Phi(\zeta))\, d\sigma_t = \int_{K_t} f\, d\mu_t.$$

It follows that

$$f(z_0) = \int_{K_t} f\, d\mu_t, \quad \forall f \in \mathcal{P}(K_t). \tag{4}$$

Here $z_0 = \Phi(0)$. It remains to observe that $\mathcal{P}(K_t) = \mathcal{O}(K_t)$ by the Oka-Weil theorem (see e.g. [19, 15]). Since $t > 0$ is arbitrary, therefore, μ_t satisfies (ii) in Lemma 1. This finishes the proof of Lemma 1.

Next, we construct our domains in $\mathbb{C}^3$ to meet the requirement of the Theorem. The construction follows from a theorem of Catlin [3]:

Proposition 2. *Let K be any compact polynomially convex set in $\mathbb{C}^n$. Then there exists a C^∞ smooth psh exhaustion function ϕ in $\mathbb{C}^n$ such that $\phi \geq 0$, $\phi^{-1}(0) = K$ and ϕ is strictly psh exactly in $\mathbb{C}^n \setminus K$.*

Let ϕ be the function obtained by applying Proposition 2 to our compact set K in $\mathbb{C}^2$. Define a family of domains Ω_δ for $\delta > 0$ in $\mathbb{C}^3$ by

$$\Omega_\delta = \{(z, w) \in \mathbb{C}^2 \times \mathbb{C} : |w|^2 + \phi(z) < \delta^2\}.$$

By Sard's theorem, for almost all $\delta > 0$, the domain Ω_δ has C^∞ boundary. Choose such a δ. By replacing ϕ by ϕ/δ^2 and w by w/δ, we may assume that $\delta = 1$ and then set $\Omega = \Omega_1$ and $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$.

Lemma 3. *The domain Ω thus constructed is a smoothly bounded pseudoconvex B-regular domain in $\mathbb{C}^3$. Moreover, $A(\overline{\Omega})|_{K \times \mathbb{T}} \subset \mathcal{O}(K \times \mathbb{T})$.*

Indeed, by Proposition 2, the set of weakly pseudoconvex boundary points of Ω is exactly $K \times \mathbb{T}$ which is B-regular by Lemma 1. It then follows that Ω is B-regular [23, 24]. Next, for any $f \in A(\overline{\Omega})$, define a sequence of functions f_n by $f_n(z, w) = f(z, nw/(n + 1))$, $n \geq 1$. Then $f_n \to f$ uniformly on $\overline{\Omega}$ and f_n is holomorphic on the domain

$$\{(z, w) \in \mathbb{C}^2 \times \mathbb{C} : \frac{n^2}{(n+1)^2}|w|^2 + \phi(z) < 1\} \supset \Omega,$$

which contains some open neighborhood of $K \times \mathbb{T}$. Thus Lemma 3 follows.

We are now in a position to finish the proof of the Theorem. Let $p_0 = (z_0, 1) \in b\Omega$, where z_0 is given in Lemma 1. We claim that p_0 is not a peak point for $\Omega \cap U$ for any neighborhood U of p_0. Suppose it is a peak point for some $\Omega \cap U$. Then there is a $f \in A(\Omega \cap U)$, such that $f(p_0) = 1$ and $|f| < 1$ on $\overline{(\Omega \cap U)} \setminus \{p_0\}$. If we set $f_0(z) = f(z, 1)$, it follows that

$$f_0(z_0) = 1, \quad |f_0(z)| < 1 \quad \forall z \in K_t \setminus \{z_0\}, \quad \text{for some } t > 0. \tag{5}$$

In view of Lemma 1 (ii), there is a nontrivial probability measure μ_t on K_t such that

$$|g(z_0)| \leq \int_{K_t} |g(z)|\, d\mu_t, \quad \forall g \in \mathcal{O}(K_t). \tag{6}$$

On the other hand, from the proof of Lemma 3, we see that f_0 can be approximated by the functions $f_n(z) =: f(z, \frac{n}{n+1})$ in $\mathcal{O}(K_t)$. Therefore, it follows from (6) that

$$|f_0(z_0)| \leq \int_{K_t} |f_0(z)|\, d\mu_t < 1. \tag{7}$$

Since μ_t is not a point mass, (7) contradicts (5). This completes the proof of the Theorem.

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