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OPERATORS α-COMMUTING WITH A COMPACT OPERATOR

VASILE LAURIC

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Abstract. In this note we update a question raised by Pearcy and Shields ('74) concerning the invariant subspace problem on Hilbert spaces.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. We shall write $\mathcal{L}(\mathcal{H})\backslash\{\lambda\}$ for the set of all operators in $\mathcal{L}(\mathcal{H})$ that are not scalar multiples of the identity operator and $\mathbb{K}$ for the ideal of compact operators in $\mathcal{L}(\mathcal{H})$. At this time the invariant subspace problem (ISP) for operators in $\mathcal{L}(\mathcal{H})$ remains unsolved. Nevertheless, serious progress on the ISP has been made by many authors at different times. One striking result, obtained by V. Lomonosov in 1973, is the following theorem (see [16], [17], and the bibliography for additional results in this direction).

Theorem 1 ([14]). *If $T \in \mathcal{L}(\mathcal{H})\backslash\{\lambda\}$ and there exists $K \in \mathbb{K}\backslash\{0\}$ such that $TK = KT$, then T has a nontrivial hyperinvariant subspace (n.h.s.).*

This result led the authors of [16] to ask in '74 whether Lomonosov's theorem above actually solves the ISP, in the following sense. Let us define

$$\mathcal{S} := \{T \in \mathcal{L}(\mathcal{H}) : \exists\, A \in \mathcal{L}(\mathcal{H})\backslash\{\lambda\}\ \exists\, K \in \mathbb{K}\backslash\{0\}\ [TA = AT \wedge AK = KA]\}.$$

Then, according to Theorem 1 above, every operator T in $\mathcal{S}$ has a nontrivial invariant subspace (n.i.s.), and thus the question was raised whether $\mathcal{S} = \mathcal{L}(\mathcal{H})$. Initially it was believed that if $\mathcal{S}$ were not all of $\mathcal{L}(\mathcal{H})$, then the shift operator M_z acting on $H^2(\mathbf{T})$, the Hardy space of square integrable functions with respect to the normalized Lebesgue measure on the unit circle $\mathbf{T}$, would be a good candidate for an operator not in the set $\mathcal{S}$. Since the commutant of the shift operator above consists of the set of all analytic Toeplitz operators, the problem whether $M_z \in \mathcal{S}$ is equivalent to the problem whether some nonscalar analytic Toeplitz operator commutes with a nonzero compact operator. This problem was eventually settled in the late '70's by Cowen [4],[5],[6] who proved that there are nonscalar analytic Toeplitz operators which commute with a nonzero compact operator, and thus that $M_z \in \mathcal{S}$.

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Shortly thereafter it was shown in [9] that $\mathcal{L}(\mathcal{H}) \neq \mathcal{S}$ by proving that the only compact operator which commutes with a nonscalar operator from the commutant of the shift operator M_z acting on a certain *weighted* $H^2(\beta)$ Hilbert space is zero.

Of course, it follows that Lomonosov's theorem stated above did not solve the ISP. In the years following the publication of [14] several generalizations of Theorem 1 were found by various authors (cf., for example, the bibliography). In particular, the following result was obtained by S. Brown (and independently by Kim-Pearcy-Shields).

Theorem 2 ([1]). *If $T \in \mathcal{L}(\mathcal{H})\backslash\{\lambda\}$ and there exists $K \in \mathbb{K}\backslash\{0\}$ such that $TK = \alpha KT$ for some complex number α, then T has a n.h.s.*

This theorem leads naturally to an "updated" Pearcy-Shields question which motivated this note. Let us define

$$\tilde{\mathcal{S}} := \{T \in \mathcal{L}(\mathcal{H}) : \exists\, A \in \mathcal{L}(\mathcal{H})\backslash\{\lambda\}\ \exists\, K \in \mathbb{K}\backslash\{0\}\ \exists\, \alpha \in \mathbf{C}\ [TA = AT \wedge AK = \alpha KA]\},$$

$$\tilde{\mathcal{T}} := \{T \in \mathcal{L}(\mathcal{H})\backslash\{\lambda\} : \exists\, K \in \mathbb{K}\backslash\{0\}\ \exists\, \alpha \in \mathbf{C}\ [TK = \alpha KT]\}.$$

Question 3. Does $\tilde{\mathcal{S}} = \mathcal{L}(\mathcal{H})$?

Question 4. Does $\tilde{\mathcal{T}} = \mathcal{L}(\mathcal{H})$?

Of course, once again, if the answer to Question 3 is affirmative, then the ISP is solved (since by Theorem 2 above every operator in $\tilde{\mathcal{S}}$ has a n.i.s.). However, one may easily check that the example furnished in [9] to show that $\mathcal{S} \neq \mathcal{L}(\mathcal{H})$ does not belong to $\tilde{\mathcal{S}}$; thus $\tilde{\mathcal{S}} \supsetneq \mathcal{S}$ and [9] does not answer Question 3.

The purpose of this note is to make a modest contribution to the above questions by showing that there are "many" operators that do not commute with a (nonzero) compact operator, but do α-commute with such an operator, and thus that the set $\tilde{\mathcal{S}}$ is very likely much larger than $\mathcal{S}$. Moreover, we show that the answer to Question 4 is "no".

2. Some Toeplitz operators

Let $L^2(\mathbf{T})$ denote the usual Hilbert space of square integrable functions on the unit circle $\mathbf{T}$ relative to normalized Lebesgue measure on $\mathbf{T}$, and let $L^\infty(\mathbf{T})$ denote the algebra of essentially bounded functions in $L^2(\mathbf{T})$. Recall that if $\phi \in L^\infty(\mathbf{T})$ and P denotes the orthogonal projection of $L^2(\mathbf{T})$ onto $H^2(\mathbf{T})$, then the Toeplitz operator T_ϕ with symbol ϕ acting on $H^2(\mathbf{T})$ is defined by $T_\phi f = P(\phi f),\quad f \in H^2(\mathbf{T})$. We consider the class of Toeplitz operators

$$\mathcal{G} := \{T_\phi : \phi(e^{it}) = ae^{-ikt} + be^{ikt},\ k \in \mathbb{N},\ |b| < |a|\}, \tag{1}$$

and we prove the following.

Theorem 5. *If $T_\phi \in \mathcal{G}$ then T_ϕ commutes with no operator in $\mathbb{K}\backslash\{0\}$, but there exists a neighborhood $\mathcal{O}_\phi$ of the origin in $\mathbf{C}$ such that for each $\alpha \in \mathcal{O}_\phi$ there exists an operator $K_\alpha \in \mathbb{K}\backslash\{0\}$ such that $T_\phi K_\alpha = \alpha K_\alpha T_\phi$.*

Before beginning the proof of the theorem we need some notation and preliminaries. For purposes of our discussion, we may assume that $a = 1$. Let us write $\{e_n(t) = e^{int} : n \in \mathbb{N}_0\}$ for the canonical orthonormal basis of $H^2(\mathbf{T})$, and also write $x \sim (x_0, x_1, \dots)$ to mean that a vector $x \in H^2(\mathbf{T})$ has the expression

$x = \sum_{n\in\mathbb{N}_0} x_n e_n$. The spectral properties of the operator T_ϕ when $k = 1$ are nicely presented in a paper of Duren [7]. We briefly recall some of them. The equation $T_\phi x = \lambda x$ is equivalent to the recurrence relations

$$-\lambda x_0 + x_1 = 0,$$
$$x_{n+1} - \lambda x_n + b x_{n-1} = 0, \qquad n \in \mathbb{N},$$

where $x \sim (x_0, x_1, \dots)$. One may solve this system of equations with $x_0 = 1$ by induction to obtain $x_n = p_n(\lambda)$, $n \in \mathbb{N}$, where $p_n(\lambda)$ is a polynomial of degree n. The number λ is an eigenvalue of T_ϕ with the corresponding eigenvector $v(\lambda) \sim (1, p_1(\lambda), \dots)$ if and only if the sequence $(1, p_1(\lambda), \dots)$ is square-summable. The function ϕ transforms each circle centered at zero and of radius $\rho > 0$ into an ellipse $\mathcal{E}_\rho$ centered at zero. We denote by $Int\,\mathcal{E}_\rho$ the bounded component of $\mathbf{C}\backslash\mathcal{E}_\rho$. Thus, if $\frac{1}{r}\,(:= |b|) < 1$, from the general theory of Toeplitz operators we know that the spectrum $\sigma(T_\phi)$ of the operator T_ϕ consists of the union of $\mathcal{E}_1$ and $Int\,\mathcal{E}_1$. Moreover, each $\lambda \in Int\,\mathcal{E}_1$ is an eigenvalue of the operator T_ϕ and $dim\,Ker(T_\phi - \lambda) = 1$. From [7, Lemma 1], $\sum_n |p_n(\lambda)|^2$ converges uniformly on each closed subset of $Int\,\mathcal{E}_1$. Furthermore, we have the following property of orthogonality which appeared first in [18] (see also [7, Lemma 2]). For each $\rho > 0$,

$$\frac{1}{2\pi}\int_{\mathcal{E}_\rho} p_n(\lambda)\overline{p_m(\lambda)}\omega(\lambda)|d\lambda| = \begin{cases} 0, & m \neq n, \\ \rho^{2(n+1)} + (r/\rho)^{2(n+1)}, & m = n, \end{cases} \tag{2}$$

where $\omega(\lambda) = |\lambda^2 - 4b|^{1/2}$. Using these facts we get that $\vee\{v(\lambda) : \lambda \in Int\,\mathcal{E}_1\} = H^2(\mathbf{T})$. Indeed, if $x \in \big(\vee\{v(\lambda) : \lambda \in Int\,\mathcal{E}_1\}\big)^{\perp}$, then $\sum_n p_n(\lambda)\overline{x}_n = 0$, where $x \sim (x_0, x_1, \dots)$, and by the two lemmas mentioned above, $x_n = 0$, $n \in \mathbb{N}_0$. Furthermore, because of the analyticity of the function $\lambda \mapsto v(\lambda)$, we have $\vee\{v(\lambda_n)\} = H^2(\mathbf{T})$ for any sequence $\{\lambda_n\}$ converging to a point λ_0 of $Int\,\mathcal{E}_1$. We will refer to this as the spanning property of $v(\lambda)$.

Proof of Theorem 5. As previously mentioned, we may assume that $a = 1$ and thus $|b| < 1$. First we consider the case when $k = 1$. According to [19, Cor. 5.14, pp. 354], there is a class of Toeplitz operators containing the class $\mathcal{G}$ such that no operator in that class commutes with a nonzero compact operator. An alternative brief proof for our class is as follows. Suppose that there exists a compact operator K such that $T_\phi K = K T_\phi$. Applying this operator equality to $v(\lambda)$, $\lambda \in Int\,\mathcal{E}_1$, we get

$$(T_\phi - \lambda)Kv(\lambda) = 0, \quad \lambda \in Int\,\mathcal{E}_1. \tag{3}$$

Since $Ker\,(T_\phi - \lambda)$ is 1-dimensional for $\lambda \in Int\,\mathcal{E}_1$, there exists a complex-valued function $\gamma(\lambda)$ such that $Kv(\lambda) = \gamma(\lambda)v(\lambda)$, $\lambda \in Int\,\mathcal{E}_1$. It can be easily seen that $\gamma(\lambda)$ is analytic on $Int\,\mathcal{E}_1$, and because $\gamma(\lambda)$ is an eigenvalue of K and K is compact, the function γ must be a constant function. Since $v(\lambda)$ has the spanning property, $\gamma \equiv 0$ and K must be zero.

We prove now the second part of the theorem. First we choose a neighborhood $\mathcal{O}_\phi$ of the origin such that for every $\alpha \in \mathcal{O}_\phi$, $\alpha\,\mathcal{E}_1 \subseteq Int\,\mathcal{E}_1$. Next we observe that it suffices to exhibit a compact operator K_α satisfying

$$K_\alpha v(\lambda) = v(\alpha\lambda), \quad \lambda \in \mathcal{O}_\phi. \tag{4}$$

Indeed, if $K_\alpha \in \mathbb{K}$ and satisfies (4), then

$$T_\phi K_\alpha v(\lambda) = T_\phi v(\alpha\lambda) = \alpha\lambda v(\alpha\lambda) = \alpha K_\alpha(\lambda v(\lambda)) = \alpha K_\alpha T_\phi v(\lambda),\ \lambda \in \mathcal{O}_\phi,$$

and since $v(\lambda)$ has the spanning property, $T_\phi K_\alpha = \alpha K_\alpha T_\phi$. Thus our problem is reduced to exhibiting a compact operator satisfying (4). We may think matricially, and suppose that the desired K_α is formally associated with a matrix $\left(k_{ij}\right)_{i,j\in\mathbb{N}_0}$ (with respect to the canonical basis $\{e_n(e^{it})\}$ of $H^2(\mathbf{T})$). Then (4) is equivalent to the system of equations

$$\sum_{j\in\mathbb{N}_0} k_{ij}\, p_j(\lambda) = p_i(\alpha\lambda), \quad i \in \mathbb{N}_0,\ \lambda \in \mathbf{C}. \tag{5}$$

But these equations are independent of one another, and we can solve them, one-at-a-time, beginning with $i = 0$ (which has solution $k_{00} = 1$ and $k_{0j} = 0,\ j \in \mathbb{N}$) and proceeding next to $i = 1$, etc. Clearly we obtain a unique matrix (k_{ij}) solving (5) with the property that for all $i \in \mathbb{N}_0,\ k_{ij} = 0$ for $j > i$. Thus it suffices to demonstrate that this matrix (k_{ij}) is the matrix of a compact operator. In fact, we will show that this matrix is square-summable, and thus that the operator K_α is a Hilbert-Schmidt operator.

Indeed, multiplying (5) by $\overline{p_j(\lambda)}\omega(\lambda)$, integrating on $\mathcal{E}_\rho$ with respect to $|d\lambda|$ and using (2), we get

$$\frac{1}{2\pi}\int_{\mathcal{E}_\rho} k_{ij} p_j(\lambda)\overline{p_j(\lambda)}\omega(\lambda)|d\lambda| = \frac{1}{2\pi}\int_{\mathcal{E}_\rho} p_i(\alpha\lambda)\overline{p_j(\lambda)}\omega(\lambda)|d\lambda|.$$

Hence,

$$k_{ij} = \frac{\frac{1}{2\pi}\int_{\mathcal{E}_\rho} p_i(\alpha\lambda)\cdot\overline{p_j(\lambda)}\cdot\omega(\lambda)\,|d\lambda|}{\rho^{2(j+1)} + (\frac{r}{\rho})^{2(j+1)}}, \qquad i,\ j \in \mathbb{N}_0.$$

Choosing $1 < \rho < r$, we get

$$|k_{ij}|^2 \leq c(r)\int_{\mathcal{E}_\rho} |p_i(\alpha\lambda)|^2 \cdot |p_j(\lambda)|^2 \cdot \omega(\lambda)^2\,|d\lambda|, \qquad i,\ j \in \mathbb{N}_0,$$

where $c(r) := \int_{\mathcal{E}_\rho} 1\,|d\lambda|$. Since $\alpha \in \mathcal{O}_\phi$, the compact set $\mathcal{E}_\rho \cup \alpha\,\mathcal{E}_\rho \subset Int\,\mathcal{E}_1$ and we may apply Lemma 1 of [7] and get that $\sum_{i,j}|k_{ij}|^2 < \infty$.

The proof for $k > 1$ consists only of noticing that the operator $T_{\phi(z^k)}$ is unitarily equivalent to the k-ampliation $T_{\phi(z)} \oplus \cdots \oplus T_{\phi(z)}$. □

Remark 1. In case ϕ is as in (1) except that $|a| < |b|$, one sees easily that T_ϕ also does not commute, but does α-commute, with some nonzero compact operator. Indeed, $T_\phi^* = T_{\overline{\phi}}$, and we can apply Theorem 5 to $T_{\overline{\phi}}$.

Remark 2. A result generalizing Theorem 5 is true for any operator $A \in \mathcal{L}(\mathcal{H})$ whose matrix has the form (with respect to some orthonormal basis $\{e_n\}_{n\in\mathbb{N}_0}$ of $\mathcal{H}$)

$$A = \begin{pmatrix} 0 & a_0 & 0 & 0 & \dots \\ b_1 & 0 & a_1 & 0 & \dots \\ 0 & b_2 & 0 & a_2 & \dots \\ 0 & 0 & b_3 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the sequences $\{a_n\}$ and $\{b_n\}$ are defined in terms of a fixed sequence $\{d_n\}$ with $d_0 = 1$, a number $c \geq 1$, and positive numbers m and M satisfying

$$m\, c^n \leq |d_n| \leq M\, c^n, \quad n \in \mathbb{N},$$

by the equations

$$b_n = b(d_{n-1}/d_n),\ n \in \mathbb{N},$$
$$a_n = (d_{n+1}/d_n),\ n \in \mathbb{N}_0,$$

where $a \neq 0$ and $|b| < 1$.

Indeed, the equation $(A - \lambda)x = 0,\ x \in \mathcal{H}$, is equivalent to to the system

$$\begin{aligned} -\lambda x_0 + a_0 x_1 &= 0; \\ a_n x_{n+1} - \lambda x_n + b_n x_{n-1} &= 0, \quad n \in \mathbb{N}, \end{aligned} \tag{6}$$

where $x = \sum_{n\in\mathbb{N}_0} x_n e_n$. Thus (6) becomes

$$-\lambda x_0 + d_1 x_1 = 0;$$
$$d_{n+1}x_{n+1} - \lambda\, d_n x_n + b\, d_{n-1}x_{n-1} = 0,\ n \in \mathbb{N}.$$

If we set $x_0 = 1$, then the above set of equations has a unique solution $x_n = q_n(\lambda)$, where $q_n(\lambda)$ is a polynomial of degree n. Setting $p_n(\lambda) := d_n\, q_n(\lambda)$, we get

$$p_{n+1}(\lambda) - \lambda p_n(\lambda) + b p_{n-1}(\lambda) = 0,\ n \in \mathbb{N}_0,$$

with $p_0(\lambda) = 1,\ p_{-1}(\lambda) := 0$. It is known (from the above preliminary) that for $|b| < 1$ and $\lambda \in Int\,\mathcal{E}_1$, the sequence $(1, p_1(\lambda), \dots)$ is square-summable, satisfies (2), and $v(\lambda) := \sum_{n\in\mathbb{N}_0} p_n(\lambda)e_n$ has the spanning property. Thus we obtain an orthogonality property for $(1, q_1(\lambda), \dots)$, namely

$$\frac{1}{2\pi}\int_{\mathcal{E}_\rho} q_n(\lambda)\overline{q_m(\lambda)}\omega(\lambda)|d\lambda| = \frac{\delta_{mn}}{|d_n|^2}\left[\rho^{2(n+1)} + (\frac{r}{\rho})^{2(n+1)}\right],$$

for each $\rho > 0$, where $r = \frac{1}{|b|}$. Therefore,

$$\int_{\mathcal{E}_\rho} q_n(\lambda)\overline{q_n(\lambda)}\omega(\lambda)|d\lambda| \leq \frac{2\pi}{A^2c^{2n}}\left[\rho^{2(n+1)} + (\frac{r}{\rho})^{2(n+1)}\right]. \tag{7}$$

Because $\sum_{n\in\mathbb{N}_0}|p_n(\lambda)|^2$ converges uniformly on each compact subset of $Int\,\mathcal{E}_1$, $\sum_{n\in\mathbb{N}_0}|q_n(\lambda)|^2$ does also, and using (7) we obtain that $\tilde{v}(\lambda) := \sum_{n\in\mathbb{N}_0} q_n(\lambda)e_n$ has the spanning property. These facts are sufficient to construct, in the same way as in Theorem 5, a nonzero compact operator which α-commutes with the operator A, and to show that A does not commute with any nonzero compact operator.

We close this note by proving the following.

Proposition 6. *If $N \in \mathcal{L}(\mathcal{H})$ is a normal operator with empty point spectrum, $\alpha \in \mathbf{C}$, and $K \in \mathbb{K}$ such that $NK = \alpha KN$, then $K = 0$ (and thus $\tilde{\mathcal{T}} \neq \mathcal{L}(\mathcal{H})$).*

Proof. Let us suppose that there exist a nonzero compact operator K and a complex number α such that $NK = \alpha KN$. Since the point spectrum of N is empty, $\alpha \neq 0$. By Fuglede-Putnam's theorem we have $N^*K = \alpha KN^*$. Since $\alpha \neq 0,\ N(K^*K) = (K^*K)N$. Since $K \neq 0,\ K^*K \neq 0$, and thus K^*K has a positive eigenvalue p_0. Because N commutes with K^*K, the corresponding finite dimensional eigenspace is invariant under N, and thus N has point spectrum, contradicting the hypothesis. □

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Department of Mathematics, Texas A&M University, College Station, Texas 77843
E-mail address: `lauric@math.tamu.edu`