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FUNCTIONS ON THE MODULI SPACE OF FLAT G_2-CONNECTIONS ON A RIEMANN SURFACE

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ABSTRACT. This paper contains $1\frac{1}{2}$ proofs that the construction of functions on the moduli space of flat G-connections from chord diagrams gives all functions in the case of the simple group G_2.

1. Introduction

By now it is well known that there are deep connections between knot theory and quantum field theory ([Wit89],[RT90],[RT91] etc.). On the other hand, a major recent development in knot theory was the invention of Vassiliev invariants and the theory of chord diagrams ([Vas90]). These employed standard techniques of topology and singularity theory, but soon it was discovered that these invariants are related to Feynman diagrams and Lie algebras (see e.g. [BN95]).

Motivated by earlier work of Goldman [Gol86] and Turaev [Tur91], chord diagrams on an arbitrary Riemann surface Σ were introduced in [AMR]. Fix a Lie groups G with invariant inner product on the Lie algebra. Functions on moduli space form a Poisson algebra whose quantization attracted a lot of attention (see for example [Ati90]). To each chord diagram D one can associate a function $f_{\rho,D}$ on the moduli space of flat G-connections over Σ (we have to colour D by representations ρ of G). It turns out that one can define a simple Poisson structure on the algebra of chord diagrams which trivially extends to coloured chord diagrams such that the map $D \mapsto f_{\rho,D}$ is a Poisson homomorphism. Since the Poisson structure on chord diagrams is independent of G this suggests the possibility of a universal quantization of moduli spaces.

In [AMR] it was shown for some groups that the map $D \mapsto f_{\rho,D}$ is surjective. This was done by reducing it to a problem about intertwiners for representations of the Lie algebra $\mathfrak{g}$ of G ([AMR], Theorem 10). In this paper we show surjectivity for the simple group G_2.

2. Notations and conventions

The Dynkin diagram will be taken to be 1 =>= 2. The standard representation $V = V_{\omega_2} = V_{0,1}$ is also a representation of $\mathfrak{b}_3$. A weight basis for V is denoted by

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$v_1, v_2, v_3, \ldots$ with v_1 being the highest weight vector. The adjoint representation is $V_{\omega_1} = V_{1,0}$.

Let $\{I_\mu\}$ be an orthonormal basis for $\mathfrak{g}_2$ with respect to the given invariant inner product. c_λ will denote the action of the quadratic Casimir $\sum I_\mu I_\mu$ on the irreducible representation V_λ.

c stands for $\sum I_\mu \otimes I_\mu$. Given a tensor product of representations $V_1 \otimes \ldots \otimes V_n$ the action of $\sum 1 \otimes \ldots \otimes 1 \otimes \overset{i}{I_\mu} \otimes 1 \otimes \ldots \otimes \overset{j}{I_\mu} \otimes \ldots \otimes 1$ is denoted by c_{ij}.

3. Proof of surjectivity

Since every irreducible representation of $\mathfrak{g}_2$ imbeds into some tensor power $V^{\otimes n}$ of the standard representation it follows from Theorem 10 of [AMR] that the map from chord diagrams to functions on moduli space is surjective provided

$$End^{\mathfrak{g}_2}\left(V^{\otimes n}\right) = \mathbb{C}\left(\{c_{ij}\}_{i<j}^{n}\right).$$

Lemma 3.1. *$End^{\mathfrak{g}_2}\left(V^{\otimes 2}\right) = \mathbb{C}\left(c\right)$.*

Proof. In the decomposition

$$V_{0,1} \bigotimes V_{0,1} = V_{0,0} \bigoplus V_{1,0} \bigoplus V_{0,1} \bigoplus V_{0,2}$$

the Casimir operator has a different eigenvalue on each summand. Therefore any intertwiner is a linear combination of $c^0, \ldots, c^3$. □

The following is an interesting consequence. Recall that $\mathfrak{g}_2 \subseteq \mathfrak{b}_3$ and hence $End^{\mathfrak{b}_3}\left(V^{\otimes n}\right) \subseteq End^{\mathfrak{g}_2}\left(V^{\otimes n}\right)$.

Corollary 3.2. *Any $\mathfrak{b}_3$-intertwiner of $V^{\otimes n}$ is generated by Casimirs for $\mathfrak{g}_2$, i.e. $End^{\mathfrak{b}_3}\left(V^{\otimes n}\right) \subseteq \mathbb{C}\left(\{c_{ij}\}_{i<j}^{n}\right)$.*

Proof. Any $\mathfrak{b}_3$-intertwiner of $V^{\otimes n}$ is a composition of intertwiners $c_{ij}^{\mathfrak{b}_3}$. These act nontrivially only in 2 factors in $V^{\otimes n}$ and the lemma implies they can be written as compositions of Casimirs. □

Lemma 3.3. *$End^{\mathfrak{g}_2}\left(V^{\otimes 3}\right) = \mathbb{C}\left(c_{12}, c_{13}, c_{23}\right)$*

Proof. We compare

$$V^{\otimes 3} = V_{0,0} \bigoplus 2V_{1,0} \bigoplus 4V_{0,1} \bigoplus 2V_{1,1} \bigoplus 3V_{0,2} \bigoplus V_{0,3} \tag{3.1}$$

with

$$V^{\otimes 3} =_{\mathfrak{b}_3} 3V_{1,0,0} \oplus 2V_{1,1,0} \oplus V_{0,0,2} \oplus V_{3,0,0}.$$

The $\mathfrak{b}_3$ modules branch as follows: $V_{1,0,0} = V_{0,1}, V_{1,1,0} = V_{1,0} \oplus V_{1,1} \oplus V_{0,2}, \Lambda^3 V = V_{0,0,2} = V_{0,0} \oplus V_{0,1} \oplus V_{0,2}, V_{3,0,0} = V_{0,3}$.

Using the fact that $c_{V^{\otimes 3}} = 3c_{\omega_1} + 2\left(c_{12} + c_{13} + c_{23}\right) \in \mathbb{C}\left(c_{12}, c_{13}, c_{23}\right)$ has distinct eigenvalues on the summands in (3.1) we can project onto any summand in

$$\begin{aligned} End^{\mathfrak{g}_2}\left(V^{\otimes 3}\right) &= End\left(V_{0,0}\right) \bigoplus End\left(2V_{1,0}\right) \bigoplus End\left(4V_{0,1}\right) \\ &\bigoplus End\left(2V_{1,1}\right) \bigoplus End\left(3V_{0,2}\right) \bigoplus End\left(V_{0,3}\right). \end{aligned}$$

The summands $V_{0,0}$ and $V_{0,3}$ pose no problem. Looking at the branching over $\mathfrak{b}_3$ we see that $2V_{1,0} \subseteq 2V_{1,1,0}$ and $2V_{1,1} \subseteq V_{1,1,0}$. Using Corollary 3.2 we get any intertwiner of $2V_{1,1,0}$ and therefore also all intertwiners in $End\left(2V_{1,0}\right) \bigoplus End\left(2V_{1,1}\right)$.

Next consider the summand $End\,(3V_{0,2})$. We have

$$3V_{0,2} \subseteq 2V_{1,1,0} \oplus \left((3V_{0,2}) \cap \Lambda^3 V\right).$$

So we can get any intertwiner in $(End\,(2V_{0,2}) \oplus (0)) \subseteq End\,(3V_{0,2})$. Now we show that the 3rd copy $V_{0,2} = (3V_{0,2}) \cap \Lambda^3 V$ is not invariant under the c_{ij}: This holds since the highest weight vector is $v_1 \wedge v_2 \wedge v_3$. If $c_{ij}\,(v_1 \wedge v_2 \wedge v_3) \in \Lambda^3 V$ this would imply $c_{12}\,(v_1 \wedge v_2 \wedge v_3) = c_{23}\,(v_1 \wedge v_2 \wedge v_3)$, but c acts differently on $v_1 \wedge v_2$ than on $v_2 \wedge v_3$. Simple linear algebra shows now that $End\,(3V_{0,2}) \subseteq \mathbb{C}\,(c_{12}, c_{13}, c_{23})$.

$End\,(4V_{0,1})$ can be dealt with similarly. □

Theorem 3.4. $End^{\mathfrak{g}_2}\,(V^{\otimes n}) = \mathbb{C}\,\{c_{ij}\}_{i<j}^n$.

Proof. It follows from [AMR], Theorem 10, and [Kup94], Theorem 3.2, that it is sufficient to show that we can obtain any $\mathfrak{g}_2$-intertwiner for $V^{\otimes 3} = V_{0,1}^{\otimes 3}$. □

4. Some explicit calculations

A few hours of computer time give the following explicit forms for the intertwiners c_{12}, c_{13}, c_{23} in a basis of eigenvectors for c_{12}, acting on $4V_{0,1}$:

$$c_{12} = \begin{pmatrix} -12 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

$$c_{13} = \begin{pmatrix} 0 & 0 & \frac{36}{7} & 0 \\ 0 & -3 & 0 & 21 \\ 2 & 0 & -6 & 24 \\ 0 & 1 & \frac{7}{2} & -7 \end{pmatrix},$$

$$c_{23} = \begin{pmatrix} 0 & 0 & -\frac{36}{7} & 0 \\ 0 & -3 & 0 & -21 \\ -2 & 0 & -6 & -24 \\ 0 & -1 & -\frac{7}{2} & -7 \end{pmatrix}.$$

Again we see that they algebraically generate all of $End\,(4V)$: Since all eigenvalues of c_{12} are different there are polynomials p_i such that $p_i\,(c_{12})\,(e_j) = \delta_{i,j} e_i$. Therefore for any nonzero entry $a_{i,j}$ of a matrix A that we can construct from we can construct $E_{i,j} = \frac{1}{a_{i,j}} p_j\,(c_{12}) \circ A \circ p_i\,(c_{12})$. So all we have to show is that for every i, j there is an A with $(A)_{i,j} \neq 0$. We can take for example $c_{13}, (c_{13})^2, (c_{13})\,(c_{23})^2$.

5. Sketch of an alternative proof

Define the following ordering on the weight lattice: $(p,q) \geq (m,n)$ iff $p+q > m+n$ or $p+q = m+n$ and $p \geq m$.

Computations suggest the following:

$$V_{0,p} \otimes V_{n,m} = V_{n,m+p} \oplus \sum_{(k,l)<(n,m+p)} V_{l,k}$$

(this might be known, but I could not find a reference).

Assuming this and the following assertion about symmetric powers,

$$S^p V_{0,1} = \sum_k V_{0,p-2k}, \tag{5.1}$$

$$S^p V_{1,0} = V_{p,0} \oplus \sum_{(k,l)<(p,0)} V_{k,l}, \tag{5.2}$$

we can give an alternative proof of Theorem 3.4, avoiding the use of [Kup94], Theorem 3.2. We will do this by induction using the above ordering on weights and employing Corollary 3.2.

The induction will start as follows: Every $\mathfrak{g}_2$ intertwiner of $nV = \oplus_n V$ is determined by its action on the highest weight vectors, thus is in fact a $\mathfrak{b}_3$-intertwiner. Consider now $nV_{1,0} : n\Lambda^2 V_{0,1} = nV_{1,0} \oplus nV_{0,1}$ So every $\mathfrak{g}_2$ intertwiner of $nV_{1,0}$ is a $\mathfrak{b}_3$ intertwiner of $n\Lambda^2 V_{0,1}$ minus an intertwiner of $nV_{0,1}$.

Note that $V_{n,m} \subseteq S^n (V_{1,0} + V_{0,1}) \otimes S^m (V_{0,1})$. According to the above decompositions the right-hand side is of the form

$$\begin{aligned} S^n (V_{1,0} + V_{0,1}) \otimes S^m (V_{0,1}) &= \left(V_{n,0} \oplus \sum_{(k,l)<(n,0)} V_{k,l} \right) \otimes \left(\sum_k V_{0,m-2k} \right) \\ &= V_{n,m} \oplus \sum_{(k,l)<(n,m)} V_{l,k}. \end{aligned}$$

Therefore any $\mathfrak{g}_2$-intertwiner of $\oplus_k V_{n,m}$ is the difference of a $\mathfrak{b}_3$-intertwiner of $kS^n \left(\Lambda^2 V_{0,1}\right) \otimes S^m (V_{0,1})$ and intertwiners of $kV_{\alpha,\beta}$ that we already constructed by induction. This finishes the sketch of the alternative proof.

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