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ON THE GANEA CONJECTURE FOR MANIFOLDS

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ABSTRACT. Using a result of Singhof, we prove that $\operatorname{cat}(M \times S^m) = \operatorname{cat} M + 1$ provided M is a connected closed PL manifold with $\dim M \leq 2 \operatorname{cat} M - 3$ and S^m is the m-sphere, $m > 0$.

Let $\operatorname{cat} X$ denote the Lusternik–Schnirelmann category of X (normalized, i.e., $\operatorname{cat} S^m = 1$). There is a long standing Ganea conjecture that $\operatorname{cat}(X \times S^m) = \operatorname{cat} X + 1$ for every connected finite CW-complex X and every $m > 0$, see [1, Problem 2]. In [2, Corollary 6.7] Singhof proved the following

Singhof's Theorem. (i) *Let M be a connected closed PL manifold such that*

$$\operatorname{cat} M \geq \frac{\dim M + m + 2}{2}.$$

Then $\operatorname{cat}(M \times S^m) = \operatorname{cat} M + 1$ *provided* $m > 0$.

(ii) *In particular, if*

$$\operatorname{cat} M \geq \frac{\dim M + 3}{2}$$

then $\operatorname{cat}(M \times S^1) = \operatorname{cat} M + 1$.

However, it seems that nobody noted that Singhof's Theorem implies a stronger result. Namely, the following theorem is valid:

Theorem. *Let M be a connected closed PL manifold such that*

$$\operatorname{cat} M \geq \frac{\dim M + 3}{2}.$$

Then $\operatorname{cat}(M \times S^m) = \operatorname{cat} M + 1$ *for every* $m > 0$.

Proof. First, we prove that $\operatorname{cat}(M \times T^r) = \operatorname{cat} M + r$ where T^r is the r-torus. We prove this by induction. For $r = 1$ this follows from (ii). Now, suppose that $\operatorname{cat}(M \times T^r) = \operatorname{cat} M + r$. Then

$$\begin{aligned}\operatorname{cat}(M \times T^r) = \operatorname{cat} M + r &\geq \frac{\dim M + 3}{2} + r \\ &\geq \frac{\dim M + r + 3}{2} = \frac{\dim(M \times T^r) + 3}{2}.\end{aligned}$$

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So, by (ii), $\operatorname{cat}(M \times T^{r+1}) = \operatorname{cat} M + r + 1$.

Now, given $m > 0$, we prove that $\operatorname{cat}(M \times S^m) = \operatorname{cat} M + 1$. Choose r such that

$$\operatorname{cat}(M \times T^r) = \operatorname{cat} M + r \geq \frac{\dim(M \times T^r) + m + 2}{2}$$

(for example, $r \gg m$). Then, by (i),

$$\operatorname{cat}(M \times T^r \times S^m) = \operatorname{cat}(M \times T^r) + 1 = \operatorname{cat} M + r + 1.$$

Now, if $\operatorname{cat}(M \times S^m) \neq \operatorname{cat} M + 1$ then $\operatorname{cat}(M \times S^m) \leq \operatorname{cat} M$. But then

$$\operatorname{cat}(M \times S^m \times T^r) \leq \operatorname{cat}(M \times S^m) + \operatorname{cat} T^r \leq \operatorname{cat} M + r.$$

This is a contradiction. □

Corollary. *For every connected closed PL manifold M there exists a natural number k such that* $\operatorname{cat}(M \times T^k \times S^m) = \operatorname{cat}(M \times T^k) + 1$ *for every $m > 0$.*

Proof. Indeed, you can find k such that

$$\operatorname{cat}(M \times T^k) \geq k \geq \frac{\dim(M \times T^k) + 3}{2}.$$

□

References

1. T. Ganea, *Some problems on numerical homotopy invariants*, Symposium in Algebraic Topology, Seattle 1971, Edited by P. Hilton (Lect. Notes in Math., vol. 249), Berlin Heidelberg New York: Springer, 1971, pp. 23–30. MR **49:**3910
2. W. Singhof, *Minimal coverings of manifolds with balls*, Manuscripta Math. **29** (1979), 385–415. MR **80k:**55012

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