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WAVELET DECOMPOSITIONS OF FOURIER MULTIPLIERS

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Abstract. We show that in terms of its weak* topology, the space of Fourier multipliers for $L^p(\mathbb{R})$, $1 < p < \infty$, can be decomposed by band-limited wavelets belonging to the Schwartz class.

1. Introduction and notation

A function $w \in L^2(\mathbb{R})$ is called a *wavelet* provided that the sequence $\{w_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ defined by $w_{j,k} \equiv 2^{j/2} w\left(2^j x - k\right)$ is an orthonormal basis for $L^2(\mathbb{R})$. It is well-known that under these circumstances $\{w_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ often serves as a basis for numerous spaces of interest in analysis. In the spirit of this theme, the present note applies the theory of wavelets to Fourier analysis *per se* by showing that, in an appropriate sense, certain wavelets belonging to the Schwartz class $\mathcal{S}$ on $\mathbb{R}$ decompose the Fourier multipliers for $L^p(\mathbb{R})$, $1 < p < \infty$. In order to state and establish our results (see §2), we begin by recalling some essential background items and fixing notation. Suppose that $1 \le p < \infty$, q is the index conjugate to p, and Γ is the dual group of a locally compact abelian group G. The space $M_p(\Gamma)$ of Fourier multipliers for $L^p(G)$ consists of the functions $\psi \in L^\infty(\Gamma)$ such that T_ψ is a continuous linear mapping of $L^p(G)$ into $L^p(G)$, where T_ψ is given by the formula $T_\psi f = \left(\psi \widehat{f}\right)^\vee$, for all $f \in L^2(\mathbb{R}) \bigcap L^p(\mathbb{R})$. $M_p(\Gamma)$ is a Banach algebra under pointwise operations and the norm $\|\psi\|_{M_p(\Gamma)} \equiv \|T_\psi\|_p$. As shown in [5, Theorem 1], when $1 < p < \infty$, $M_p(\Gamma)$ can be identified (via a surjective linear isometry) with the dual space of $A_p(G)$. The Banach space $A_p(G)$ is composed of the functions f which can be written in the form

$$f = \sum_{n=1}^{\infty} f_n * g_n, \tag{1}$$

where $\{f_n\}_{n=1}^\infty \subseteq L^p(G)$, $\{g_n\}_{n=1}^\infty \subseteq L^q(G)$, and $\sum_{n=1}^\infty \|f_n\|_p \|g_n\|_q < \infty$. The norm $\|f\|_{A_p(G)}$ is the infimum of the sums $\sum_{n=1}^\infty \|f_n\|_p \|g_n\|_q$ corresponding to the representations for f in (1). By starting with an arbitrary representation (1) for $f \in A_p(G)$, we can specify the duality relationship between $M_p(\Gamma)$ and $A_p(G)$ by writing $\psi[f] = \sum_{n=1}^\infty \big((T_\psi f_n) * g_n\big)(0)$. Furthermore, if $F \in L^1(\Gamma)$, and $g(x) \equiv \int_\Gamma F(\gamma)\gamma(x)\, d\gamma$, then $g \in A_p(G)$, and $\psi[g] = \int_\Gamma \psi(\gamma) F(\gamma)\, d\gamma$. (It follows

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from this, by a standard Hahn-Banach argument, that $L^1(\Gamma)^\frown$ is norm dense in $A_p(G)$–a fact also shown in the course of [5, proof of Theorem 1].) The wavelet decompositions in $M_p(\mathbb{R})$ described below will be formulated in terms of the weak* topology $\sigma(M_p(\mathbb{R}), A_p(\mathbb{R}))$.

Let Ω be the class of wavelets $w \in \mathcal{S}$ such that: (i) the support of $\widehat{w}(t) \equiv \int_{\mathbb{R}} w(x)e^{-ixt}\,dx$ has the form $\{t \in \mathbb{R} : \pi - \varepsilon \le |t| \le 2(\pi+\varepsilon)\}$, where ε belongs to $(0, \pi/3]$; and (ii) $|\widehat{w}|$ is an even function. The most general wavelet in $L^2(\mathbb{R})$ satisfying (i) and (ii) has been characterized in [4, Theorem (3.1)]. The study of the class Ω has its origins in [6], where the constructions furnished an element of Ω such that ε in (i) has the value $\frac{\pi}{3}$. Subsequently it was shown in [3, §§1-3] that for each ε belonging to $(0, \pi/3]$ there is a corresponding wavelet $w \in \Omega$ such that (i) holds. Now suppose that $\Lambda \in \Omega$. By virtue of [2, Theorem 1.3] the wavelet basis $\{\Lambda_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ arises from a multiresolution analysis which has an associated scaling function $\Delta \in \mathcal{S}$ such that $\widehat{\Delta}$ is compactly supported. As usual, we choose Δ so that $\int_{\mathbb{R}} \Delta(x)\,dx = 1$. (This notation for $\Lambda \in \Omega$ and $\Delta \in \mathcal{S}$ will remain in effect henceforth.) The self-adjoint projection operators $\{E_j\}_{j=-\infty}^{\infty}$ corresponding to the multiresolution analysis have the following familiar description:

(2)

$$(E_j f)(x) = 2^j \int_{\mathbb{R}} K\left(2^j x, 2^j y\right) f(y)\,dy = \sum_{k=-\infty}^{\infty} \langle f, \Delta_{j,k}\rangle\, \Delta_{j,k}(x), \quad \text{for all } x \in \mathbb{R},$$

where $\Delta_{j,k}(x) \equiv 2^{j/2}\Delta\left(2^j x - k\right)$, $K(x,y) \equiv \sum_{k=-\infty}^{\infty} \Delta_{0,k}(x)\overline{\Delta_{0,k}}(y)$, and $\langle f, \Delta_{j,k}\rangle = \int_{\mathbb{R}} f(y)\overline{\Delta_{j,k}}(y)\,dy$. Since

$$\sum_{k=-\infty}^{\infty} |\Delta_{0,k}(x)\Delta_{0,k}(y)| \le C_m(1+|x-y|)^{-m}, \quad \text{for all } x \in \mathbb{R},\ , y \in \mathbb{R},\ \text{and } m \in \mathbb{N},$$

the scope of (2) is not limited to the functions f belonging to the initial space $L^2(\mathbb{R})$: for $1 \le p \le \infty$, (2) also defines E_j as a continuous linear mapping of $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ such that $E_j^2 = E_j$. Analogous assertions are valid for the idempotent operators $\{D_j\}_{j=-\infty}^{\infty}$ defined on $L^p(\mathbb{R})$, $1 \le p \le \infty$, by $D_j = E_{j+1} - E_j$. Specifically, we have

(3)

$$(D_j f)(x) = 2^j \int_{\mathbb{R}} \widetilde{K}\left(2^j x, 2^j y\right) f(y)\,dy = \sum_{k=-\infty}^{\infty} \langle f, \Lambda_{j,k}\rangle\, \Lambda_{j,k}(x), \quad \text{for all } x \in \mathbb{R},$$

where $\widetilde{K}(x,y) \equiv \sum_{k=-\infty}^{\infty} \Lambda_{0,k}(x)\overline{\Lambda_{0,k}}(y)$ and

$$\sum_{k=-\infty}^{\infty} |\Lambda_{0,k}(x)\Lambda_{0,k}(y)| \le C_m(1+|x-y|)^{-m} \quad \text{for all } x \in \mathbb{R},\ y \in \mathbb{R},\ \text{and } m \in \mathbb{N}.$$

We shall also benefit from the relations of scale: $E_j = \delta_{2^j} E_0 \delta_{2^{-j}}$ and $D_j = \delta_{2^j} D_0 \delta_{2^{-j}}$, where $(\delta_\alpha f)(x) \equiv f(\alpha x)$. Notice that $D_m D_n = 0$ for $m \ne n$, since $j \le k$ implies $E_j E_k = E_k E_j = E_j$ on $L^p(\mathbb{R})$, $1 \le p \le \infty$.

The following theorem furnishes the starting point for the links between wavelets and multipliers developed in §2.

Theorem 1 ([1, Theorem (5.6)]). *Suppose that $h \in L^1(\mathbb{R})$ is continuous on $\mathbb{R}$, $B \in (0,\infty)$, the support of $\widehat{h}$ is a subset of $[-B,B]$, $\widehat{h}$ is absolutely continuous on $\mathbb{R}$, and the derivative of $\widehat{h}$ belongs to $L^2(\mathbb{R})$. If $1 \le p < \infty$ and $\phi \in M_p(\mathbb{Z})$, then $\mathcal{W}_{\phi,h} \in M_p(\mathbb{R})$, where $\mathcal{W}_{\phi,h}(x) = \sum_{k=-\infty}^{\infty} \phi(k)h(x-k)$ for all $x \in \mathbb{R}$. Moreover*

$$\|\mathcal{W}_{\phi,h}\|_{M_p(\mathbb{R})} \le 2^{N+(\max\{p,q\})^{-1}} \delta(h) \|\phi\|_{M_p(\mathbb{Z})},$$

where N is the least non-negative integer such that $B2^{-N} \le \pi/2$, and

$$\delta(h) = \sup_{x\in\mathbb{R}} \sum_{k=-\infty}^{\infty} |h(x-k)| < \infty.$$

$\mathcal{W}_{\phi,h}$ can be extended to $\mathbb{C}$ as an entire function of exponential type.

2. Wavelet decompositions in $M_p(\mathbb{R})$

Our main results are stated in the following two theorems. Observe beforehand that since $M_p(\mathbb{R}) \subseteq L^\infty(\mathbb{R})$, the operators D_j and E_j, $j \in \mathbb{Z}$, map $M_p(\mathbb{R})$ into $L^\infty(\mathbb{R})$.

Theorem 2. *Suppose that $1 < p < \infty$ and $\Lambda \in \Omega$. Then for each $j \in \mathbb{Z}$, D_j and E_j are continuous idempotent linear transformations of $Mp(\mathbb{R})$ into $M_p(\mathbb{R})$. The image $D_j(M_p(\mathbb{R}))$ is the weak*-closed subspace of $M_p(\mathbb{R})$ generated by $\{\Lambda_{j,k}\}_{k=-\infty}^{\infty}$. If $\psi \in M_p(\mathbb{R})$, then $\sum_{k=-N}^{N} \langle \psi, \Lambda_{j,k}\rangle \Lambda_{j,k}$ converges in the weak* topology $\sigma(M_p(\mathbb{R}), A_p(\mathbb{R}))$ to $D_j\psi$, as $N \to \infty$. If, in addition, $\psi(t) \to 0$ as $|t| \to \infty$, then $\sum_{j=-N}^{N} D_j\psi$ converges weak* and almost everywhere to ψ, as $N \to \infty$.*

Theorem 3. *Suppose that $1 < p < \infty$ and $\Lambda \in \Omega$. Then the weak*-closed subspace of $M_p(\mathbb{R})$ generated by $\{\Lambda_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ is $M_p(\mathbb{R})$.*

Proof of Theorem 2. Let $\psi \in M_p(\mathbb{R})$ and put $\Lambda^\star(t) \equiv \overline{\Lambda(-t)}$. Then, with the aid of De Leeuw's Restriction Theorem for Multipliers, we find that

$$\left\|\{\langle \psi, \Lambda_{0,k}\rangle\}_{k=-\infty}^{\infty}\right\|_{M_p(\mathbb{Z})} = \|(\Lambda^\star * \psi)|\,\mathbb{Z}\|_{M_p(\mathbb{Z})} \le \|\Lambda\|_1 \|\psi\|_{M_p(\mathbb{R})}.$$

In view of Theorem 1 this implies that D_0 is a continuous linear mapping of $M_p(\mathbb{R})$ into $M_p(\mathbb{R})$ (and similarly so is E_0). If $F \in L^1(\mathbb{R})^\frown$, it follows by dominated convergence that $\left(\sum_{k=-N}^{N} \langle \psi, \Lambda_{0,k}\rangle \Lambda_{0,k}\right)[F] \to (D_0\psi)[F]$, as $N \to \infty$. For $N \in \mathbb{N}$, let $\chi_N : \mathbb{Z} \to \mathbb{C}$ be the characteristic function of $[-N,N] \cap \mathbb{Z}$. The M. Riesz Theorem, together with an application of Theorem 1 to $\left\{\langle \psi, \Lambda_{0,k}\rangle \chi_N(k)\right\}_{k=-\infty}^{\infty} \in M_p(\mathbb{Z})$, shows that

$$\sup_{N\in\mathbb{N}} \|\textstyle\sum_{k=-N}^{N} \langle \psi, \Lambda_{0,k}\rangle \Lambda_{0,k}\|_{M_p(\mathbb{R})} < \infty.$$

Since $L^1(\mathbb{R})^\frown$ is dense in $A_p(\mathbb{R})$, it is clear from the foregoing observations that, as $N \to \infty$,

$$\sum_{k=-N}^{N} \langle \psi, \Lambda_{0,k}\rangle \Lambda_{0,k} \to D_0\psi, \quad \text{in the weak}^* \text{ topology of } M_p(\mathbb{R}).$$

Conversely, let ϕ belong to the weak*-closed subspace Q of $M_p(\mathbb{R})$ generated by $\{\Lambda_{0,k}\}_{k=-\infty}^{\infty}$, and put $\Phi = \phi - D_0\phi \in Q$. Since

$$\Psi\left[\,\overline{\widehat{f}}\,\right] = \langle \Psi, f\rangle, \quad \text{for } \Psi \in M_p(\mathbb{R}),\ f \in L^1(\mathbb{R}), \tag{4}$$

it is clear from the orthonormality of $\{\Lambda_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ that $\langle \phi, \Lambda_{j,k}\rangle = 0$, for $j \in \mathbb{Z}\setminus\{0\}$, $k \in \mathbb{Z}$. Applying this result to $\Phi \in Q$ in place of ϕ, we readily infer that $\langle \Phi, \Lambda_{j,k}\rangle = 0$ for all $j \in \mathbb{Z}$ and all $k \in \mathbb{Z}$. In view of the well-known fact that $\{\Lambda_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}}$ generates $H^1(\mathbb{R})$, this shows that Φ is a constant. Because $\Phi \in Q$, another application of (4) gives $\langle \Phi, \Delta\rangle = 0$. Hence the constant value of Φ is 0, and so $\phi = D_0\phi$. After a change of scale from the level $j = 0$ to the general level $j \in \mathbb{Z}$, it remains only to establish the last assertion of the theorem. For this purpose suppose first that ψ is an arbitrary element of $M_p(\mathbb{R})$. The reasoning in [7, p. 33], with obvious modifications, shows that $(E_j\psi)(x) \to \psi(x)$ as $j \to \infty$, for every x in the Lebesgue set of ψ. If we further assume that $\lim_{|t|\to\infty}\psi(t) = 0$, then another elementary argument using the kernel K in (2) shows that $\|E_j\psi\|_\infty \to 0$, as $j \to -\infty$. Since $\sum_{j=-N}^{N} D_j = E_{N+1} - E_{-N}$, the required a.e. convergence is evident, and the required weak* convergence follows with the aid of a dominated convergence argument. □

Proof of Theorem 3. Suppose, to the contrary, that there is $f = \sum_{n=1}^{\infty} f_n * g_n \in A_p(\mathbb{R})$ such that $\|f\|_{A_p(\mathbb{R})} = 1$ and for all $j \in \mathbb{Z}$, $k \in \mathbb{Z}$

$$0 = \Lambda_{j,k}[f] = \textstyle\sum_{n=1}^{\infty} \left((\Lambda_{j,k})^\vee * (f_n * g_n)\right)(0) = (2\pi)^{-1}\int_{\mathbb{R}} f(t)\widehat{\Lambda}_{j,k}(t)\,dt.$$

This implies that $f \equiv 0$ by virtue of the reasoning used for the final stage in the proof of [4, Theorem (3.1)], and consequently we have arrived at a contradiction. □

(5) **Remark.** With obvious modifications, the reasoning used to establish Theorem 2 also shows that if $1 < p < \infty$, $j \in \mathbb{Z}$, and $\psi \in M_p(\mathbb{R})$, then $\sum_{k=-N}^{N} \langle \psi, \Delta_{j,k}\rangle \Delta_{j,k}$ converges weak* to $E_j\psi$, as $N \to \infty$.

The next result shows that the operators E_j and D_j on $M_p(\mathbb{R})$ preserve weak* convergence.

Theorem 4. *Suppose that $1 < p < \infty$ and $\Lambda \in \Omega$. Then for $j \in \mathbb{Z}$ each of the operators E_j and D_j on $M_p(\mathbb{R})$ is the adjoint of a corresponding continuous linear operator mapping $A_p(\mathbb{R})$ into $A_p(\mathbb{R})$.*

Proof. Denote by $C_0^\infty(\mathbb{R})$ the dense linear subspace of $A_p(\mathbb{R})$ consisting of all $f \in C^\infty(\mathbb{R})$ such that f has compact support. For $f \in C_0^\infty(\mathbb{R})$ and $k \in \mathbb{Z}$, we have

$$\Delta_{0,k}[f] = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(t)\Delta(t-k)\,dt.$$

Since $\widehat{f} \in \mathcal{S}$ and $\Delta \in \mathcal{S}$, standard estimates on the right-hand side of this equation prove that the sequence $\{\Delta_{0,k}[f]\}_{k=-\infty}^{\infty}$ is rapidly decreasing–that is, for each $m \in \mathbb{N}$, there is a constant C_m such that

$$|\,\Delta_{0,k}[f]\,| \le \frac{C_m}{|k|^m}, \quad \text{for all } k \in \mathbb{Z}\setminus\{0\}.$$

Next suppose that $F \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$. Let F_t denote the translate of F by t. By performing simple calculations which use the duality relationship between $A_p(\mathbb{R})$

and $M_p(\mathbb{R})$, we see that

$$\left\| (F_t)^{\wedge} \right\|_{A_p(\mathbb{R})} = \left\| F^{\wedge} \right\|_{A_p(\mathbb{R})} .$$

It follows immediately from the foregoing discussion that if $f \in C_0^\infty(\mathbb{R})$, then the series

$$\sum_{k=-\infty}^{\infty} (\Delta_{0,k}[f]) \, \overline{\widehat{\Delta}}_{0,k}$$

converges absolutely in the space $A_p(\mathbb{R})$ to some vector g. Hence for each $\psi \in M_p(\mathbb{R})$, we infer with the aid of (4) that

$$\psi[g] = \sum_{k=-\infty}^{\infty} (\Delta_{0,k}[f]) \, \langle \psi, \Delta_{0,k} \rangle \, .$$

Applying Remark (5) (in the case $j = 0$) to this, we get $\psi[g] = (E_0\psi)\,[f]$, and consequently $\|g\|_{A_p(\mathbb{R})} \leq K \|f\|_{A_p(\mathbb{R})}$, where K denotes the norm of E_0 as a continuous linear map of $M_p(\mathbb{R})$ into $M_p(\mathbb{R})$. Invoking the density of $C_0^\infty(\mathbb{R})$ in $A_p(\mathbb{R})$, we can now readily deduce that the operator E_0 on $M_p(\mathbb{R})$ is the adjoint of a bounded linear transformation mapping $A_p(\mathbb{R})$ into $A_p(\mathbb{R})$. The assertion of the theorem regarding E_j, $j \in \mathbb{Z}$, follows at once by a change of scale, and then furnishes the desired conclusion for $D_j = E_{j+1} - E_j$. □

In closing we observe that methods based on Theorem 1 similar to those used in the demonstration of Theorem 2 can be combined with the Closed Graph Theorem to give the following result.

Proposition. *If $\Lambda \in \Omega$ and $1 \leq p < \infty$, then D_0 is an idempotent bounded linear transformation of $M_p(\mathbb{R})$ into $M_p(\mathbb{R})$. The mapping $\phi \in M_p(\mathbb{Z}) \mapsto \mathcal{W}_{\phi,\Lambda}$ is an injective bicontinuous linear transformation of $M_p(\mathbb{Z})$ onto $D_0\,(M_p(\mathbb{R}))$.*

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