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EXISTENCE OF BADE FUNCTIONALS FOR COMPLETE BOOLEAN ALGEBRAS OF PROJECTIONS IN FRÉCHET SPACES

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ABSTRACT. A classical result of W. Bade states that if $\mathcal{M}$ is any $\sigma-$complete Boolean algebra of projections in an arbitrary Banach space X then, for every $x_0 \in X$, there exists an element x' (called a Bade functional for x_0 with respect to $\mathcal{M}$) in the dual space X', with the following two properties: (i) $M \mapsto \langle Mx_0, x'\rangle$ is non-negative on $\mathcal{M}$ and, (ii) $Mx_0 = 0$ whenever $M \in \mathcal{M}$ satisfies $\langle Mx_0, x'\rangle = 0$. It is shown that a Fréchet space X has this property if and only if it does not contain an isomorphic copy of the sequence space $\omega = \mathbb{C}^{\mathbb{N}}$.

A Boolean algebra $\mathcal{M}$ of selfadjoint projections in a Hilbert space H has the property that for every $x_0 \in H$ the inner product $\langle Ex_0, x_0\rangle$, with $E \in \mathcal{M}$, is non-negative and vanishes only if $Ex_0 = 0$. A satisfactory extension to the Banach space setting of this useful property of the inner product in Hilbert spaces is the following remarkable result of W. Bade, [1, Theorem 3.1].

Theorem 1. *Let $\mathcal{M}$ be a $\sigma-$complete Boolean algebra of projections in a Banach space X. Then, for each $x_0 \in X$, there exists a continuous linear functional $x' \in X'$ (called a Bade functional for x_0 with respect to $\mathcal{M}$) with the properties*

(i) $\langle Mx_0, x'\rangle \geq 0$, *for all $M \in \mathcal{M}$, and*
(ii) *if $\langle Mx_0, x'\rangle = 0$ for some $M \in \mathcal{M}$, then $Mx_0 = 0$.*

Theorem 1 fails to be true in the non-normable setting, even in Fréchet (locally convex) spaces. Indeed, the Boolean algebra $\mathcal{M}$ generated by the co-ordinate projections in the Fréchet space ω of all complex sequences, equipped with the topology of co-ordinatewise convergence, is $\sigma-$complete but the element $x_0 = (1, 1, \dots)$ cannot have any Bade functional with respect to $\mathcal{M}$ since the continuous dual space ω' consists of all complex sequences which have only finitely many non-zero terms.

Let $\mathcal{M}$ be a $\sigma-$complete Boolean algebra of projections in a locally convex Hausdorff space X. Then $\mathcal{M}$ is said to have *Property-*(B) if, for every $x_0 \in X$, there exists $x' \in X'$ satisfying (i) and (ii) of Theorem 1. The space X is said to have the *Bade property* if every $\sigma-$complete Boolean algebra of projections in X has Property-(B). Bade's classical theorem above asserts that every Banach space has the Bade property. The above example shows that Fréchet spaces in general do

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not have the Bade property; it turns out that this example is essentially a paradigm. The aim of this note is to establish the following result.

Theorem 2. *Let X be a Fréchet space. Then X has the Bade property if and only if it does not contain an isomorphic copy of ω.*

1. Preliminaries

A Fréchet space X is said to have *Rybakov's property*, [5], if for every $\sigma-$additive, $X-$valued vector measure ν (defined on a $\sigma-$algebra of sets Σ) there exists $x' \in X'$ such that $\nu \ll |\langle \nu, x' \rangle|$ (i.e. $\nu(F) = 0$ for every measurable set $F \subseteq E$ whenever $|\langle \nu, x' \rangle|(E) = 0$). Here $\langle \nu, x' \rangle$ denotes the complex measure $E \mapsto \langle \nu(E), x' \rangle$ and $|\langle \nu, x' \rangle|$ denotes the total variation measure of $\langle \nu, x' \rangle$.

Proposition 1 ([5, Theorem 2.2])**.** *For a **real** Fréchet space X the following conditions are equivalent.*

(i) *X admits a continuous norm.*
(ii) *Every convex, weakly compact subset of X is the closed convex hull of its exposed points.*
(iii) *X has Rybakov's property.*

We will require the following extension of this result to Fréchet spaces over $\mathbb{C}$.

Proposition 2. *For a Fréchet space X the following conditions are equivalent.*

(i) *X contains no isomorphic copy of ω.*
(ii) *X admits a continuous norm.*
(iii) *X has Rybakov's property.*

Proof. The equivalence of (i) and (ii) is well known; see [2] or [6, Theorem 7.2.7], for example.

(ii)$\Longrightarrow$(iii). Let $X_{\mathbb{R}}$ denote X considered as a linear space over $\mathbb{R}$, in which case $X_{\mathbb{R}}$ also admits a continuous norm (the same one that X does). Let $\nu : \Sigma \longrightarrow X$ be a vector measure, meaning that $\nu(A_n) \longrightarrow 0$ in X whenever $A_n \downarrow \phi$ in Σ. But then also $\nu(A_n) \longrightarrow 0$ in $X_{\mathbb{R}}$ and so ν is still a vector measure when considered as taking its values in $X_{\mathbb{R}}$. By Proposition 1 there exists a continuous linear functional $x' : X_{\mathbb{R}} \longrightarrow \mathbb{R}$ such that $\nu \ll |\langle \nu, x' \rangle|$. Define $\xi' : X \longrightarrow \mathbb{C}$ by

$$\langle x, \xi' \rangle = \langle x, x' \rangle - i\langle ix, x' \rangle, \qquad x \in X,$$

in which case $\xi' \in X'$ and satisfies

$$\langle \nu(E), \xi' \rangle = \langle \nu(E), x' \rangle - i\langle i\nu(E), x' \rangle, \qquad E \in \Sigma.$$

Accordingly, if $E \in \Sigma$ is a set such that $\langle \nu(F), \xi' \rangle = 0$ for all $F \in \Sigma$ with $F \subseteq E$, then also $\mathrm{Re}(\langle \nu(F), \xi' \rangle) = \langle \nu(F), x' \rangle = 0$ for all $F \in \Sigma$ with $F \subseteq E$. That is, $|\langle \nu, x' \rangle|(E) = 0$ whenever $|\langle \nu, \xi' \rangle|(E) = 0$ showing that $|\langle \nu, x' \rangle| \ll |\langle \nu, \xi' \rangle|$. Accordingly, $\nu \ll |\langle \nu, \xi' \rangle|$. Since ν is an arbitrary $X-$valued measure it follows that X has Rybakov's property.

(iii)$\Longrightarrow$ (ii). We show that $X_{\mathbb{R}}$ has Rybakov's property whenever X does; the conclusion then follows from Proposition 1 since any continuous norm φ on $X_{\mathbb{R}}$ induces a continuous norm $\tilde{\varphi}$ on X via the formula $\tilde{\varphi}(x) = \sup\{\varphi(\alpha x) : \alpha \in \mathbb{C},\ |\alpha| = 1\}$ for $x \in X$. So, let $\nu : \Sigma \longrightarrow X_{\mathbb{R}}$ be a vector measure. By a similar argument as in the proof of (ii)$\Longrightarrow$ (iii) it follows that ν is also a vector measure when considered as being $X-$valued. By assumption there is $x' \in X'$ satisfying $\nu \ll |\langle \nu, x' \rangle|$. Define elements $\xi_1', \xi_2' \in X_{\mathbb{R}}'$ by $\langle x, \xi_1' \rangle = \mathrm{Re}\,(\langle x, x' \rangle)$ and $\langle x, \xi_2' \rangle = \mathrm{Im}\,(\langle x, x' \rangle)$, for each $x \in X_{\mathbb{R}}$. Then $\mu_j = \langle \nu, \xi_j' \rangle : \Sigma \longrightarrow \mathbb{R}$ is a signed measure, for each $j \in \{1, 2\}$. By [3, IX Lemma 2.1] there exists $t \in (0, 1)$ such that

$$\mu_j \ll |t\mu_1 + (1-t)\mu_2| = |\langle \nu, z' \rangle|, \qquad j \in \{1, 2\},$$

where $z' = t\xi_1' + (1-t)\xi_2'$ is then an element of $X_{\mathbb{R}}'$. Suppose that $|\langle \nu, z' \rangle|(E) = 0$, that is, E is μ_j-null, for each $j \in \{1, 2\}$. This is equivalent to $\mu_j(F) = \langle \nu(F), \xi_j' \rangle = 0$ for every $j \in \{1, 2\}$ and $F \in \Sigma$ with $F \subseteq E$. That is, $\mathrm{Re}\,(\langle \nu(F), x' \rangle) = 0 = \mathrm{Im}\,(\langle \nu(F), x' \rangle)$, for all $F \in \Sigma$ with $F \subseteq E$, which is equivalent to $\langle \nu(F), x' \rangle = 0$, for all $F \in \Sigma$ with $F \subseteq E$. That is, $|\langle \nu, x' \rangle|(E) = 0$. So, we have shown that $|\langle \nu, x' \rangle| \ll |\langle \nu, z' \rangle|$ and hence, $\nu \ll |\langle \nu, z' \rangle|$. Since $z' \in X_{\mathbb{R}}'$ it follows that $X_{\mathbb{R}}$ has Rybakov's property. $\square$

Combining Proposition 2 with Theorem 2 (yet to be proved) gives the following consequence.

Corollary 2.1. *For a Fréchet space X the following conditions are equivalent.*

(i) *X does not contain an isomorphic copy of ω.*
(ii) *X admits a continuous norm.*
(iii) *X has Rybakov's property.*
(iv) *X has the Bade property.*

The space of all continuous linear operators of a locally convex Hausdorff space X into itself is denoted by $L(X)$. A Boolean algebra of projections $\mathcal{M} \subseteq L(X)$, with unit always the identity operator I in X, is said to be *complete* ($\sigma-$*complete*), in the sense of Bade, if it is complete ($\sigma-$ complete) as an abstract Boolean algebra and, whenever $\{M_\alpha\}_\alpha$ is a family (sequence) from $\mathcal{M}$, then

$$(\wedge_\alpha M_\alpha)X = \cap_\alpha M_\alpha X \quad \text{and} \quad (v_\alpha M_\alpha)X = \overline{\mathrm{sp}}\{\cup_\alpha M_\alpha X\}.$$

Of course, the partial order in $\mathcal{M}$ is range inclusion, i.e. $M \le N$ if $MX \subseteq NX$. If X is a Fréchet space, then $\mathcal{M}$ is necessarily an equicontinuous part of $L(X)$, [9, Proposition 1.2]. Moreover, it is known that $\mathcal{M} = P(\Sigma)$ is the range of some spectral measure $P : \Sigma \longrightarrow L(X)$ defined on a $\sigma-$algebra of sets Σ of some set Ω, [9, p. 299]. By a *spectral measure* we mean that P is $\sigma-$additive with respect to the strong operator topology in $L(X)$, that $P(\Omega) = I$, and that P is multiplicative (i.e. $P(E \cap F) = P(E)P(F)$ for all $E, F \in \Sigma$).

Given a vector $x_0 \in X$ the *cyclic space* $\mathcal{M}[x_0]$ generated by x_0 with respect to $\mathcal{M}$ is the closure of the linear span of $\{Mx_0; M \in \mathcal{M}\}$. The $X-$valued (actually $\mathcal{M}[x_0]-$valued) vector measure $Px_0 : \Sigma \longrightarrow X$ is defined by $Px_0 : E \mapsto P(E)x_0$, for each $E \in \Sigma$. Associated with the vector measure Px_0 is its (locally convex Hausdorff) space $L^1(Px_0)$ of (equivalence classes of) Px_0-integrable functions $f : \Omega \longrightarrow \mathbb{C}$, equipped with the topology of convergence in mean; see [4, Section 1], for example. For the notion of a *closed* vector measure we also refer to [4, Section 1]. The *integration map* $I_{Px_0} : L^1(Px_0) \longrightarrow X$ is defined by $f \mapsto \int_\Omega f dPx_0$, for each $f \in L^1(Px_0)$.

Proposition 3 ([4, Proposition 2.1])**.** *Let X be a quasicomplete locally convex Hausdorff space such that $L(X)$ is sequentially complete for the strong operator topology. Let $\mathcal{M} \subseteq L(X)$ be an equicontinuous, complete Boolean algebra of projections, displayed as the range of a closed, equicontinuous spectral measure $P : \Sigma \longrightarrow$*

$L(X)$. *Then, for each* $x_0 \in X$, *the integration map* $I_{Px_0} : L^1(Px_0) \longrightarrow \mathcal{M}[x_0]$ *induces on the cyclic space* $\mathcal{M}[x_0]$ *the structure of a Dedekind complete, locally solid (complex) Riesz space with Lebesgue topology in which* x_0 *is a weak order unit. The absolute value of an element* $\int_\Omega f dPx_0 \in \mathcal{M}[x_0]$ *is the element* $\int_\Omega |f| dPx_0 \in \mathcal{M}[x_0]$. *Moreover,* I_{Px_0} *is a Riesz isomorphism as well as a topological isomorphism and the absolute value mapping on* $\mathcal{M}[x_0]$ *is continuous.*

2. Proof of Theorem 2

Let $\mathcal{M} \subseteq L(X)$ be any $\sigma-$complete Boolean algebra of projections. Since $\mathcal{M}$ is equicontinuous, [9, Proposition 1.2], the closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ in $L(X)$, with respect to the strong operator topology, is an equicontinuous complete Boolean algebra, [9, Proposition 3.17]. Since $\mathcal{M}[x_0] = \overline{\mathcal{M}}[x_0]$ it is clear from the definition of the Bade property that it suffices to show that every *complete* Boolean algebra of projections in $L(X)$ has Property-(B) if and only if X does not contain an isomorphic copy of ω.

Suppose that X does not contain a copy of ω in which case it admits a continuous norm (by Proposition 2). Let $\mathcal{M} \subseteq L(X)$ be any complete Boolean algebra, necessarily equicontinuous. Fix $x_0 \in X$. Let $P : \Sigma \longrightarrow L(X)$ be a closed, equicontinuous spectral measure such that $\mathcal{M} = P(\Sigma)$, [9, p.299]. Since X admits a continuous norm so does $\mathcal{M}[x_0]$. So, to establish that X has the Bade property it suffices to establish the following result.

Proposition 4. *Let* X *be a quasicomplete locally convex Hausdorff space such that* $L(X)$ *is sequentially complete for the strong operator topology. Let* $\mathcal{M} \subseteq L(X)$ *be a complete, equicontinuous Boolean algebra of projections. Suppose* $x_0 \in X$ *has the property that the cyclic space* $\mathcal{M}[x_0]$ *is a Fréchet space for the relative topology from* X. *The following statements are equivalent.*

(i) x_0 *has a Bade functional with respect to* $\mathcal{M}$.
(ii) $\mathcal{M}[x_0]$ *admits a continuous norm.*

Proof. (i)$\Longrightarrow$ (ii) Suppose that $x' \in X'$ is a Bade functional for x_0. By Proposition 3 the restriction of x' to $\mathcal{M}[x_0]$ acts via the formula

$$\langle \int_\Omega f dPx_0, x' \rangle = \int_\Omega f d\langle Px_0, x' \rangle, \qquad f \in L^1(Px_0),$$

where $P : \Sigma \longrightarrow L(X)$ is a closed, equicontinuous spectral measure such that $\mathcal{M} = P(\Sigma)$. Define $q : \mathcal{M}[x_0] \longrightarrow [0, \infty)$ by $q(\int_\Omega f dPx_0) = \int_\Omega |f| d\langle Px_0, x' \rangle$, for each $f \in L^1(Px_0)$. By property (i) of Bade functionals (cf. Theorem 1) it is clear that q does indeed take non-negative values and, by the properties of integration with respect to a non-negative measure, it is clear that q is a seminorm on $\mathcal{M}[x_0]$. Since

$$\int_\Omega |f| d\langle Px_0, x' \rangle = \langle \int_\Omega |f| dPx_0, x' \rangle = \langle | \int_\Omega f dPx_0 |_{\mathcal{M}[x_0]}, x' \rangle,$$

for each $f \in L^1(Px_0)$, where $|\cdot|_{\mathcal{M}[x_0]}$ is the absolute value in the Riesz space $\mathcal{M}[x_0]$, necessarily continuous (cf. Proposition 3), it follows that q is continuous on $\mathcal{M}[x_0]$. Finally, to see that q is actually a norm, suppose that $q(\int_\Omega f dPx_0) = 0$, i.e. $|f|$ is $\langle Px_0, x' \rangle-$null. By property (ii) of Bade functionals (cf. Theorem 1) and the fact that $\mathcal{M} = P(\Sigma)$, there is a set $E \in \Sigma$ such that $f(w) = 0$ for $w \notin E$ and $P(E)x_0 = 0$. By multiplicativity of P it follows that $P(F)x_0 = 0$ for every $F \in \Sigma$

such that $F \subseteq E$ i.e. E is a Px_0-null set and so f is Px_0-null. Accordingly, $\int_\Omega f dPx_0 = 0$ which establishes that q is a norm.

(ii)$\Longrightarrow$ (i) Suppose that $\mathcal{M}[x_0]$ admits a continuous norm. By the Rybakov property for the vector measure $Px_0 : \Sigma \longrightarrow \mathcal{M}[x_0]$ (cf. Proposition 2) there exists $x' \in \mathcal{M}[x_0]'$ (which can also be considered as an element of X' by the Hahn-Banach theorem) satisfying $Px_0 \ll |\langle Px_0, x'\rangle|$. By the Radon-Nikodym theorem there exists a $\Sigma-$measurable function $\varphi : \Omega \longrightarrow \mathbb{C}$ with $|\varphi(w)| = 1$, for $\langle Px_0, x'\rangle$ - a.e. point $w \in \Omega$, such that

$$|\langle Px_0, x'\rangle|(E) = \int_E \varphi d\langle Px_0, x'\rangle, \qquad E \in \Sigma.$$

Define φ to be zero on the $\langle Px_0, x'\rangle-$null set for which $|\varphi| \neq 1$, in which case φ is a bounded, $\Sigma-$measurable function. Accordingly, φ is $P-$integrable and $T = \int_\Omega \varphi dP \in L(X)$, [9, p.300]. Let $z' = T'x'$ where T' denotes the dual operator to T. Then $z' \in X'$ satisfies

$$\langle P(E)x_0, z'\rangle = \int_E \varphi d\langle Px_0, x'\rangle = |\langle Px_0, x'\rangle|(E), \qquad E \in \Sigma.$$

It follows that $\langle Px_0, z'\rangle$ is a non-negative measure with the property that $P(E)x_0 = 0$ whenever $\langle P(E)x_0, z'\rangle = 0$, i.e. z' is a Bade functional for x_0 with respect to $\mathcal{M}$.

Conversely, suppose that X does contain an isomorphic copy of ω. Then it contains a complemented copy of ω, [6, Theorem 7.2.7], and so we have the direct decomposition $X = Y \oplus \omega$. Let $\Sigma = 2^{\mathbb{N}}$ and define a spectral measure $Q : \Sigma \longrightarrow L(\omega)$ by $Q(E) : \psi \mapsto \chi_E \psi$ (co-ordinatewise multiplication), for each $\psi \in \omega$ and $E \in \Sigma$. Then define a spectral measure $P : \Sigma \longrightarrow L(X)$ by $P(E) = 0_Y \oplus Q(E)$, for each $E \in \Sigma$, where 0_Y is the zero operator on Y. As noted in the introduction the vector $u_0 = (1, 1, \dots)$ of ω has no Bade functional with respect to $Q(\Sigma)$ from which it follows easily (since $X' = Y' \oplus \omega'$) that $x_0 = 0 \oplus u_0$ is an element of X which has no Bade functional with respect to the complete Boolean algebra $\mathcal{M} = P(\Sigma) \subseteq L(X)$. Accordingly, X does not have the Bade property. Equivalently, if X has the Bade property, then it cannot contain a copy of ω. The proof of Theorem 2 is thereby complete. □

Remark 1. (i) Since $\mathcal{M}[x_0]$ is a Banach space whenever X is a Banach space, it follows that Proposition 4 applies in a Banach space X to *every* $x_0 \in X$. Accordingly, Proposition 4 contains the classical Bade functional theorem. An examination of the proof of (ii)$\Longrightarrow$ (i) in Proposition 4 shows that it is quite a different proof (perhaps even more transparent) than W. Bade's original proof given in [1, Theorem 3.1]. Of course, it is based on Rybakov's theorem which, even in the Banach space setting, was not available until 1970; see [8].

(ii) The proof of (i)$\Longrightarrow$ (ii) in Proposition 4 applies in general; it does *not* require $\mathcal{M}[x_0]$ to be a Fréchet space. This was noted in [9, p.315], with a slightly different proof (based on order properties rather than vector integration), where the full statement of Proposition 4 is also alluded to, but without a proof. Whatever proof the author had in mind there it could not have been that given here since, again, Rybakov's theorem was not available at that time. □

A Boolean algebra of projections $\mathcal{M}$ acting in a locally convex Hausdorff space X is said to be *cyclic* if there exists $x_0 \in X$ such that $X = \mathcal{M}[x_0]$.

Proposition 5. *Let X be a Fréchet space which does not contain a copy of ω and let $\mathcal{M} \subseteq L(X)$ be a cyclic, $\sigma-$complete Boolean algebra of projections. Then there exists a continuous norm, $\|\cdot\|$, on X such that $\mathcal{M} \subseteq L(X_{\|\cdot\|})$, where $X_{\|\cdot\|}$ denotes X equipped with the norm topology induced by $\|\cdot\|$. Moreover, $\|M\| \leq 1$ for every $M \in \mathcal{M}$, and $\mathcal{M}$ is $\sigma-$complete considered as a Boolean algebra in $L(X_{\|\cdot\|})$.*

Proof. Let $P : \Sigma \longrightarrow L(X)$ be a spectral measure such that $\mathcal{M} = P(\Sigma)$. By hypothesis there exists $x_0 \in X$ such that $X = \mathcal{M}[x_0]$. By Proposition 3 the integration map $I_{Px_0} : L^1(Px_0) \longrightarrow X$ is a Riesz space and bicontinuous topological isomorphism which induces the absolute value map given by $|y| = \int_\Omega |f| dPx_0$ on X if $y = \int_\Omega f dPx_0$. Let $x' \in X'$ be a Bade functional for x_0 with respect to $\mathcal{M}$. It was seen in the proof of Proposition 4 that

$$\|I_{Px_0}(f)\| = \int_\Omega |f| d\langle Px_0, x'\rangle, \qquad f \in L^1(Px_0),$$

is then a continuous norm on X.

Fix $M = P(E)$ in $\mathcal{M}$. Given $x = I_{Px_0}(f)$ in X, for some unique $f \in L^1(Px_0)$, it follows from basic properties of spectral integrals (cf. [4]) that

$$Mx = P(E)x = P(E) \int_\Omega f dPx_0 = \int_\Omega f\chi_E dPx_0 = I_{Px_0}(f\chi_E).$$

These equalities show that

$$\|Mx\| = \|I_{Px_0}(f\chi_E)\| = \int_\Omega |f\chi_E| d\langle Px_0, x'\rangle \leq \int_\Omega |f| d\langle Px_0, x'\rangle = \|x\|.$$

That is, $\|M\| \leq 1$ and, in particular, $M \in L(X_{\|\cdot\|})$.

To show that $\mathcal{M}$ is a $\sigma-$complete Boolean algebra when considered in $L(X_{\|\cdot\|})$ it suffices to check that $P : \Sigma \longrightarrow L(X_{\|\cdot\|})$ is $\sigma-$additive for the strong operator topology. But, suppose that $\{A_n\}_{n=1}^\infty \subseteq \Sigma$ is a sequence decreasing to ϕ. Fix $x \in X$. Then $P(A_n)x \longrightarrow 0$, as $n \longrightarrow \infty$, in the topology of X. Since $\|\cdot\| : X \longrightarrow [0,\infty)$ is continuous also $\|P(A_n)x\| \longrightarrow 0$, as $n \longrightarrow \infty$. This establishes that P is $\sigma-$additive in $L(X_{\|\cdot\|})$ for the strong operator topology. □

Proposition 6. *Let X be the strict inductive limit of an increasing sequence of Fréchet spaces $\{X_n\}_{n=1}^\infty$ such that no $X_n, n = 1, 2, \dots,$ contains an isomorphic copy of ω. Then every equicontinuous, $\sigma-$complete Boolean algebra of projections $\mathcal{M} \subseteq L(X)$ has Property-(B).*

Proof. Fix $x_0 \in X$. The equicontinuity of $\mathcal{M}$ implies that $B(x_0) = \{Mx_0; M \in \mathcal{M}\}$ is a bounded set in X. Accordingly, there exists a positive integer n such that $B(x_0) \subseteq X_n$, [7, p.223]. Since the relative topology induced on X_n by X is precisely the given Fréchet space topology of X_n, [7, p.222], it follows that the closure of the linear span of $B(x_0)$ taken in X_n with respect to the metric topology coincides with $\mathcal{M}[x_0]$ formed in X. The conclusion now follows from Proposition 4. □

Remark 2. Let X be as in the statement of Proposition 6. Then every spectral measure $P : \Sigma \longrightarrow L(X)$ has Property-(B). This is immediate from Proposition 6 since the barrelledness of X, [7, p.368], and the boundedness of $\mathcal{M} = P(\Sigma)$ in $L(X)$, [9, p.300], imply that the $\sigma-$complete Boolean algebra $\mathcal{M}$ is necessarily equicontinuous. □

The final result, which is a simple consequence of Theorem 2, shows that there also exist extensive classes of *non-metrizable* lcH-spaces with the Bade property.

Proposition 7. *Let Y be a Fréchet space and X denote Y equipped with its weak topology $\sigma(Y, Y')$. Then X has the Bade property iff Y does not contain a copy of ω.*

Proof. Since Y has its Mackey topology, [7, p.263], it is known that $L(X) = L(Y)$ as vector spaces, [7, p.262]. Moreover, since the closure of a linear subspace of Y with respect to the metric (= Mackey) topology coincides with the $\sigma(Y, Y')-$closure, it follows from the definition that a Boolean algebra of projections $\mathcal{M}$ is $\sigma-$complete considered as a part of $L(X)$ iff it is $\sigma-$complete considered as a part of $L(Y)$. Using these observations and the fact that $X' = Y'$ it is routine to check that Y has the Bade property iff X has the Bade property. The desired conclusion then follows from Theorem 2. □

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