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ON REDUCIBILITY OF SEMIGROUPS OF COMPACT QUASINILPOTENT OPERATORS

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Abstract. The following generalization of Lomonosov's invariant subspace theorem is proved. Let $\mathcal{S}$ be a multiplicative semigroup of compact operators on a Banach space such that $\hat{r}(S_1, \dots, S_n) = 0$ for every finite subset $\{S_1, \dots, S_n\}$ of $\mathcal{S}$, where $\hat{r}$ denotes the Rota-Strang spectral radius. Then $\mathcal{S}$ is reducible.

This result implies that the following assertions are equivalent:

(A) For each infinite-dimensional complex Hilbert space $\mathcal{H}$, every semigroup of compact quasinilpotent operators on $\mathcal{H}$ is reducible.

(B) For every complex Hilbert space $\mathcal{H}$, for every semigroup of compact quasinilpotent operators on $\mathcal{H}$, and for every finite subset $\{S_1, \dots, S_n\}$ of $\mathcal{S}$ it holds that $\hat{r}(S_1, \dots, S_n) = 0$.

The question whether the assertion (A) is true was considered by Nordgren, Radjavi and Rosenthal in 1984, and it seems to be still open.

Let X be a (real or complex) Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. Let $(S_1, \dots, S_n)$ be an n-tuple of operators of $\mathcal{B}(X)$. The Rota-Strang spectral radius $\hat{r}(S_1, \dots, S_n)$ is defined by

$$\hat{r}(S_1, \dots, S_n) = \lim_{m \to \infty} \max\{\|S_{k_1} S_{k_2} \cdots S_{k_m}\|^{1/m} : 1 \le k_i \le n, 1 \le i \le m\} .$$

In [7] it is shown that this limit always exists. Similarly, we introduce the *local spectral radius* of n-tuple $(S_1, \dots, S_n)$ at a vector $x \in X$ by the formula

$$\hat{r}(S_1, \cdots, S_n; x) = \limsup_{m \to \infty} \max\{\|S_{k_1} S_{k_2} \cdots S_{k_m} x\|^{1/m} : 1 \le k_i \le n, 1 \le i \le m\} .$$

Let $\mathcal{C}$ be a collection of operators of $\mathcal{B}(X)$. We say that $\mathcal{C}$ is *reducible* if there exists a closed subspace of X, other than $\{0\}$ and X, which is invariant under every member of $\mathcal{C}$. The collection $\mathcal{C}$ is *irreducible* if it is not reducible. If there exists even a maximal subspace chain (i.e., a maximal totally ordered set of closed subspaces) whose elements are invariant under every member of $\mathcal{C}$, then $\mathcal{C}$ is said to be *simultaneously triangularizable*. Let us define the number $\hat{r}(\mathcal{C}) \in [0, \infty]$ by

$$\hat{r}(\mathcal{C}) = \sup\{\hat{r}(S_1, \dots, S_n) : n \in \mathbb{N}, S_1, \dots, S_n \in \mathcal{C}\} .$$

Similarly, we set, for any $x \in X$,

$$\hat{r}(\mathcal{C}; x) = \sup\{\hat{r}(S_1, \dots, S_n; x) : n \in \mathbb{N}, S_1, \dots, S_n \in \mathcal{C}\} .$$

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The proof of our first result is based on the famous Lomonosov-Hilden technique; see e.g. [4].

Theorem 1. *Let X be a Banach space of dimension greater than one, and let $\mathcal{S}$ be a multiplicative semigroup in $\mathcal{B}(X)$ containing a compact operator K. If $\hat{r}(\mathcal{S}K; x_0) = 0$ and $Kx_0 \neq 0$ for some vector $x_0 \in X$, then $\mathcal{S}$ is reducible.*

Proof. There is no loss of generality in assuming that $\|K\| = 1$, since we can assume that $\mathcal{S} = \mathbb{R}^+\mathcal{S}$. Replacing x_0 by λx_0 for an appropriate scalar $\lambda > 0$, we can also assume that $\|x_0\| > 1$ and $\|Kx_0\| > 1$. Let $U = \{x \in X : \|x - x_0\| \leq 1\}$ be the closed unit ball centered at x_0. Obviously, 0 is not in U nor in the closure $\overline{K(U)}$ of $K(U)$.

Let $\mathcal{A}$ be the linear manifold in $\mathcal{B}(X)$ generated by $\mathcal{S}$. Since $\mathcal{S}$ is a semigroup, $\mathcal{A}$ is equal to the subalgebra of $\mathcal{B}(X)$ generated by $\mathcal{S}$. For each fixed $y \in X$ the set

$$L_y = \{Ay : \ A \in \mathcal{A}\}$$

is a linear manifold in X which is invariant under every member of $\mathcal{A}$. If $L_y = \{0\}$ for some $y \neq 0$, then the one-dimensional subspace generated by y is invariant under every member of $\mathcal{S}$. So, we may assume that $L_y \neq \{0\}$ for all $y \neq 0$. We shall show that there exists $y \neq 0$ such that L_y is not dense in X. This will prove the theorem. Assume on the contrary that L_y is dense in X for all $y \neq 0$. Then for each $y \neq 0$ there is an $A \in \mathcal{A}$ such that $Ay \in \operatorname{int} U$, where $\operatorname{int} U$ denotes the interior of U. We therefore have

$$\overline{K(U)} \subseteq X \setminus \{0\} \subseteq \bigcup_{A \in \mathcal{A}} A^{-1}(\operatorname{int} U) \ .$$

Since $\overline{K(U)}$ is a compact set in X, there exists a finite set $\{A_1, \dots, A_n\} \subseteq \mathcal{A}$ such that

$$\overline{K(U)} \subseteq \bigcup_{i=1}^{n} A_i^{-1}(\operatorname{int} U) \ .$$

Therefore, it follows from $Kx_0 \in K(U)$ that $Kx_0 \in A_{i_1}^{-1}(U)$ for some $i_1 \in \{1, \dots, n\}$, and hence $A_{i_1}Kx_0 \in U$. Then $KA_{i_1}Kx_0 \in K(U)$ implies that $KA_{i_1}Kx_0 \in A_{i_2}^{-1}(U)$ for some i_2, and so $A_{i_2}KA_{i_1}Kx_0 \in U$. Proceeding with this "ping-pong" of Hilden's, after m steps we obtain an integer $i_m \in \{1, 2, \dots, n\}$ such that

$$A_{i_m}KA_{i_{m-1}}K \cdots A_{i_1}Kx_0 \in U \ .$$

Clearly there exist S_1, S_2, ..., $S_N \in \mathcal{S}$ and scalars $c_j^{(i)}$, $i = 1, \dots, n$, $j = 1, \dots, N$, such that

$$A_i = \sum_{j=1}^{N} c_j^{(i)} \, S_j$$

for all $i = 1, \dots, n$. Set $c = \max\{|c_j^{(i)}| : i = 1, \dots, n, j = 1, \dots, N\}$. Putting $B_i = A_iK$ for $i = 1, \dots, n$, we then have

$$\|B_{i_m}B_{i_{m-1}} \cdots B_{i_1}x_0\| \leq c^m N^m \ \max\{\|(S_{k_m}K) \cdots (S_{k_1}K)x_0\| : 1 \leq k_i \leq N\}.$$

Since $\hat{r}(S_1K, \dots, S_NK; x_0) = 0$, we conclude that

$$\lim_{m \to \infty} \|B_{i_m}B_{i_{m-1}} \cdots B_{i_1}x_0\|^{1/m} = 0 \ ,$$

which implies that

$$\lim_{m\to\infty} \|B_{i_m} B_{i_{m-1}} \cdots B_{i_1} x_0\| = 0 \ ,$$

and so $0 \in U$. This contradiction completes the proof of the theorem. $\square$

In [2] an irreducible semigroup of nilpotent operators of index two is constructed. Let us use that semigroup to show that the compactness hypothesis of Theorem 1 cannot be omitted.

Example 2. If A is a $k \times k$ matrix, then let N_A be the $2k \times 2k$ matrix

$$\begin{bmatrix} A & -A \\ A & -A \end{bmatrix} ,$$

and let T_A be the direct sum of denumerably many copies of N_A. We regard T_A as an operator on a separable Hilbert space l^2.

For each $i \in \mathbb{N}$ let $\mathcal{S}_i$ denote the linear space of all operators T_A as A ranges over all $2^{i-1} \times 2^{i-1}$ matrices, and let $\mathcal{S}$ be the union of all $\mathcal{S}_i$. It is shown in [2] that $\mathcal{S}$ is an irreducible semigroup of nilpotent operators of index two. In order to show that $\hat{r}(\mathcal{S}) = 0$, let $(S_1, \dots, S_n)$ be an n-tuple of non-zero operators of $\mathcal{S}$. Let $i(S_p)$ $(1 \le p \le n)$ denote the uniquely determined number i such that S_p belongs to $\mathcal{S}_i$. We claim that

$$S_{k_1} S_{k_2} \cdots S_{k_m} = 0$$

for all $m \ge 2^n$ and for all $1 \le k_i \le n$. Indeed, there exist $1 \le p < q \le m$ such that $k_p = k_q$ and $i(S_{k_j}) < i(S_{k_p})$ for each $p < j < q$. Using the facts that $\mathcal{S}_i \, \mathcal{S}_j \subseteq \mathcal{S}_i$ and $\mathcal{S}_j \, \mathcal{S}_i \subseteq \mathcal{S}_i$ for all $i > j$ and that $\mathcal{S}_i \, \mathcal{S}_i = \{0\}$ for all i, we now conclude that $S_{k_p} \cdots S_{k_q} = 0$, which proves the claim. It follows that $\hat{r}(S_1, \dots, S_n) = 0$, so that $\hat{r}(\mathcal{S}) = 0$.

A recent extension of Rota's theorem [3, Corollary 1] gives the following result that will be needed in the sequel.

Proposition 3. *Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{C}$ be a collection of compact quasinilpotent operators on $\mathcal{H}$ which is simultaneously triangularizable. Then $\hat{r}(\mathcal{C}) = 0$.*

Proof. Let $(S_1, \dots, S_n)$ be an n-tuple of operators of $\mathcal{C}$, and let $\epsilon > 0$. By [3, Corollary 1] there exists a positive invertible operator T such that $\|T^{-1} S_k T\| < \epsilon$ for all $k = 1, 2, \dots, n$. We then have

$$\|S_{k_1} \cdots S_{k_m}\| \le \|T\| \, \|T^{-1} S_{k_1} T\| \cdots \|T^{-1} S_{k_m} T\| \, \|T^{-1}\| < \|T\| \, \|T^{-1}\| \, \epsilon^m$$

for every $m \in \mathbb{N}$ and every $k_1, \dots, k_m \in \{1, \dots, n\}$. It follows that $\hat{r}(S_1, \dots, S_n) \le \epsilon$. Therefore, $\hat{r}(S_1, \dots, S_n) = 0$, and so $\hat{r}(\mathcal{C}) = 0$. $\square$

Let $\mathcal{H}$ denote an arbitrary complex Hilbert space, and let $\mathbf{S}(\mathcal{H})$ be the collection of all multiplicative semigroups of compact quasinilpotent operators on $\mathcal{H}$.

Theorem 4. *The following assertions are equivalent:*

(a) *For each $\mathcal{H}$ of dimension greater than one, every semigroup $\mathcal{S} \in \mathbf{S}(\mathcal{H})$ is reducible.*

(b) *For each $\mathcal{H}$, every semigroup $\mathcal{S} \in \mathbf{S}(\mathcal{H})$ is simultaneously triangularizable.*

(c) $\hat{r}(\mathcal{S}) = 0$ *for every $\mathcal{H}$ and for every $\mathcal{S} \in \mathbf{S}(\mathcal{H})$.*

(d) $\hat{r}(\mathcal{S}; x_0) = 0$ *for every $\mathcal{H}$, for every $\mathcal{S} \in \mathbf{S}(\mathcal{H})$, and for every $x_0 \in \mathcal{H}$.*

(e) *For each* $\mathcal{H}$ *and for each* $\mathcal{S} \in \mathbf{S}(\mathcal{H})$ *there exists a non-zero vector* $x_0 \in \mathcal{H}$ *such that* $\hat{r}(\mathcal{S}; x_0) = 0$.

Proof. The idea of the proof that (a) implies (b) is well-known. Since the proof is (for example) a slight modification of the proof of Corollary 1 in [5], we will omit it. By Proposition 3, (b) implies (c). It is easy to show that (c) implies (d) and that (d) implies (e).

Assume that (e) holds. Given $\mathcal{H}$ of dimension greater than one and $\mathcal{S} \in \mathbf{S}(\mathcal{H})$, there exists a non-zero vector $x_0 \in \mathcal{H}$ such that $\hat{r}(\mathcal{S}; x_0) = 0$. If $Sx_0 = 0$ for all $S \in \mathcal{S}$, then the one-dimensional subspace generated by x_0 is invariant under every member of $\mathcal{S}$, so that $\mathcal{S}$ is reducible. Otherwise, there is a $K \in \mathcal{S}$ such that $Kx_0 \neq 0$. Since $\hat{r}(\mathcal{S}K; x_0) = 0$, $\mathcal{S}$ is reducible by Theorem 1, and so (a) holds. ☐

Note that the problem (a) of the last result was studied in [5], where it is shown that every semigroup of trace class quasinilpotent operators is reducible and consequently simultaneously triangularizable.

We conclude by mentioning that the assertion (c) of Theorem 4 is not true if the compactness hypothesis is dropped. Namely, Guinand [1] constructed two weighted shifts T_1 and T_2 on l^2 of norm 1 such that every word in $\{T_1, T_2\}$ is nilpotent of index three. In other words, the semigroup $\mathcal{S}$ generated by $\{T_1, T_2\}$ consists of nilpotent operators on l^2. It was shown in [6, example] that $\hat{r}(T_1, T_2) = 1$, which implies that $\hat{r}(\mathcal{S}) = 1$.

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