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THE EXISTENCE OF POSITIVE SOLUTIONS FOR THE ONE-DIMENSIONAL p-LAPLACIAN

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Abstract. In this paper we study the existence of positive solutions of the equation $(g(u'))' + a(t)f(u) = 0$, where $g(v) = |v|^{p-2}v$, $p > 1$, subject to nonlinear boundary conditions. We show the existence of at least one positive solution by a simple application of a Fixed Point Theorem in cones and the Arzela-Ascoli Theorem.

1. Introduction

In a recent paper [2], Erbe and Wang considered the boundary value problem

$$\begin{cases} u'' + a(t)f(u) = 0, & 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, & \gamma u(1) + \delta u'(1) = 0, \end{cases} \tag{1.1}$$

under the following three assumptions:

$$\alpha, \beta, \gamma, \delta \geq 0 \text{ and } \rho := \alpha\gamma + \alpha\delta + \beta\gamma > 0. \tag{A1}$$

$$f \in C([0, +\infty), [0, +\infty)). \tag{A2}$$

$$a \in C([0,1], [0, +\infty)) \text{ and } a(t) \not\equiv 0 \text{ on any subinterval of } [0,1]. \tag{A3}$$

They obtained the following existence result:

Theorem 1. *Assume* (A1)–(A3) *hold. Then the boundary value problem* (1.1) *has at least one positive solution in the case*

(i) $f_0 = 0$ *and* $f_\infty = +\infty$, *or*
(ii) $f_0 = +\infty$ *and* $f_\infty = 0$,

where

$$f_0 := \lim_{u \downarrow 0} \frac{f(u)}{u}, \qquad f_\infty := \lim_{u \uparrow +\infty} \frac{f(u)}{u}.$$

The proof of Theorem 1 is based on the following Fixed Point Theorem due to Krasnoselskii [3].

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Theorem 2 ([1], [3]). *Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$\Phi\colon K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|\Phi u\| \le \|u\| \ \forall u \in K \cap \partial\Omega_1$, *and* $\|\Phi u\| \ge \|u\| \ \forall u \in K \cap \partial\Omega_2$; *or*
(ii) $\|\Phi u\| \ge \|u\| \ \forall u \in K \cap \partial\Omega_1$, *and* $\|\Phi u\| \le \|u\| \ \forall u \in K \cap \partial\Omega_2$.

Then Φ has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

In another recent paper [4], Yang and Fan investigated the boundary value problem

$$\begin{cases} (g(u'))' + u^a = 0, \qquad 0 < t < 1,\ a \in (0, p-1) \cup (p-1, +\infty), \\ u(0) = 0, \quad u'(1) = 0, \end{cases} \tag{1.2}$$

where $g(v) := |v|^{p-2}v$, $p > 1$, and demonstrated that the problem (1.2) has at least one positive solution, by applying Theorem 2.

Motivated by the results mentioned above, in this paper we study the existence of positive solutions of the quasilinear differential equation

$$(g(u'))' + a(t)f(u) = 0, \qquad 0 < t < 1, \tag{1.3}$$

subject to one of the following three pairs of boundary conditions:

$$u(0) - B_0(u'(0)) = 0, \quad u(1) + B_1(u'(1)) = 0, \tag{1.4$_\mathrm{a}$}$$

$$u(0) - B_0(u'(0)) = 0, \quad u'(1) = 0, \tag{1.4$_\mathrm{b}$}$$

$$u'(0) = 0, \quad u(1) + B_1(u'(1)) = 0, \tag{1.4$_\mathrm{c}$}$$

where $g(v) := |v|^{p-2}v$, $p > 1$, and hence $(g(u'))'$ is the one-dimensional p-Laplacian.

The following hypotheses are adopted throughout this paper:

(H1) $B_0(v)$ and $B_1(v)$ are both nondecreasing, continuous, odd functions defined on $(-\infty, +\infty)$ and at least one of them satisfies the condition that there exists $b > 0$ such that

$$0 \le B_j(v) \le bv \quad \text{for all } v \ge 0,\ j = 1 \text{ or } 2. \tag{1.5}$$

(H2) $f(u)$ is a nonnegative, lower semi-continuous function defined on $[0, +\infty)$. Moreover, it has only a finite number of discontinuity points of the first kind in each compact subinterval of $[0, +\infty)$.

(H3) $a(t)$ is a nonnegative measurable function defined on $(0,1)$ and satisfies the following conditions:

$$0 < \int_0^{1/2} G\left(\int_s^{1/2} a(t)dt\right) ds + \int_{1/2}^1 G\left(\int_{1/2}^s a(t)dt\right) ds < +\infty, \tag{1.6$_\mathrm{a}$}$$

if $B_0(v) \equiv 0$ and $B_1(v) \equiv 0$;

$$0 < \int_0^1 G\left(\int_s^1 a(t)dt\right) ds < +\infty, \tag{1.6$_\mathrm{b}$}$$

if $B_0(v) \equiv 0$ and $B_1(v) \not\equiv 0$ (or $u'(1) = 0$);

$$0 < \int_0^1 G\left(\int_0^s a(t)dt\right) ds < +\infty, \tag{1.6$_\mathrm{c}$}$$

if $B_1(v) \equiv 0$ and $B_0(v) \not\equiv 0$ (or $u'(1) = 0$); and

$$0 < \int_0^1 a(t)dt < +\infty, \tag{1.6$_\mathrm{d}$}$$

if $B_0(v) \not\equiv 0$ (or $u'(0) = 0$) and $B_1(v) \equiv 0$ (or $u'(0) = 0$), where $G(w) = |w|^{1/(p-1)} \operatorname{sgn} w$ is the inverse function to $g(v)$.

When $p = 2$, (1.6$_\mathrm{a}$), (1.6$_\mathrm{b}$), and (1.6$_\mathrm{c}$) can respectively be written as

$$0 < \int_0^{1/2} ta(t)dt + \int_{1/2}^1 (1-t)a(t)dt < +\infty,$$
$$0 < \int_0^1 ta(t)dt < +\infty,$$
$$0 < \int_0^1 (1-t)a(t) < +\infty,$$

which more clearly show that $a(t)$ is allowed to have singularity at the endpoints of $(0, 1)$. In addition, (H3) allows $a(t) \equiv 0$ on some subintervals of $(0, 1)$. For example, the function

$$a(t) = \begin{cases} t^{-\alpha}, & 0 < t < 1/8; 1 < \alpha < p, \\ 0, & 1/8 \le t < 1, \end{cases}$$

satisfies (H3) provided $B_0(v) \equiv 0$.

It is obvious that (1.1) and (1.2) are both particular cases of the boundary value problem (1.3)–(1.4). The aim of this paper is to extend and improve the existence results in [2], [4].

By a positive solution of the boundary value problem (1.3)–(1.4), we mean a function $u(t)$ satisfying the following conditions:

(i) $u \in C[0, 1] \cap C^1(0, 1)$ if $B_0(v) \equiv 0$ and $B_1(v) \equiv 0$;
$u \in C[0, 1] \cap C^1(0, 1]$, if $B_0(v) \equiv 0$ and $B_1(v) \not\equiv 0$ (or $u'(1) = 0$);
$u \in C[0, 1] \cap C^1[0, 1)$, if $B_1(v) \equiv 0$ and $B_0(v) \not\equiv 0$ (or $u'(0) = 0$);
$u \in C^1[0, 1]$, if $B_0(v) \not\equiv 0$ (or $u'(0) = 0$) and $B_1(v) \not\equiv 0$ (or $u'(1) = 0$).

(ii) $u(t) > 0$ for all $t \in (0, 1)$ and satisfies boundary conditions in (1.3).

(iii) $g(u'(t))$ is locally absolutely continuous in $(0, 1)$ and the equality

$$[g(u'(t))]' = -a(t)f(u(t))$$

holds almost everywhere in $(0, 1)$.

It is clear that $u \equiv 0$ is a solution of (1.3)–(1.4) when $f(0) = 0$ and each solution of (1.3)–(1.4) is a nonnegative concave function defined in $[0, 1]$. However, the concavity of solutions has not been employed in both [2] and [4].

The main result of this paper is as follows.

Theorem 3. *Assume* (H1)–(H3) *hold. Then the boundary value problem* (1.3)–(1.4) *has at least one positive solution in the case*

(i) $f_0 = 0$ *and* $f_\infty = +\infty$, *or*

(ii) $f_0 = +\infty$ *and* $f_\infty = 0$;

where

$$f_0 := \lim_{u \downarrow 0} \frac{f(u)}{u^{p-1}}, \qquad f_\infty := \lim_{u \uparrow +\infty} \frac{f(u)}{u^{p-1}}.$$

Clearly, Theorem 3 is an extension and improvement of the existence results in [2], [4].

Our arguments involve the use of the concavity and integral representation of solutions, Theorem 2, and the Arzela-Ascoli Theorem.

2. Proof of Theorem 3

It follows from (1.6) that there exists $\delta \in (0, 1/2)$ such that

$$0 < \int_\delta^{1-\delta} a(t)dt < +\infty, \tag{2.1}$$

and hence the function

$$y(x) := \int_\delta^x G\left(\int_s^x a(t)dt\right) ds + \int_x^{1-\delta} G\left(\int_x^s a(t)dt\right) ds, \qquad \delta \le x \le 1-\delta,$$

is continuous and positive on $[\delta, 1-\delta]$. In the sequel δ always satisfies (2.1).

Let K be the cone in $C[0,1]$ given by

$$K := \{u \in C[0,1];\ u(t) \text{ is a nonnegative concave function}\}.$$

Lemma 1. *Let $u \in K$ and $\delta \in (0, 1/2)$. Then*

$$u(t) \ge \begin{cases} \|u\| t/\sigma, & 0 \le t \le \sigma, \\ \|u\|(1-t)/(1-\sigma), & \sigma \le t \le 1, \end{cases} \quad \text{if } 0 < \sigma < 1, \tag{2.2$_\mathrm{a}$}$$

$$u(t) \ge \|u\| t, \qquad 0 \le t \le 1,\ \text{if } \sigma = 1, \tag{2.2$_\mathrm{b}$}$$

$$u(t) \ge \|u\|(1-t), \qquad 0 \le t \le 1,\ \text{if } \sigma = 0, \tag{2.2$_\mathrm{c}$}$$

$$u(t) \ge \delta \|u\| \quad \text{for all } t \in [\delta, 1-\delta]. \tag{2.3}$$

Here $\|u\| := \sup\{|u(t)|; 0 \le t \le 1\}$ *and* $\sigma \in [0,1]$ *such that* $u(\sigma) = \|u\|$.

Proof. The lemma follows from the concavity of $u(t)$ on $[0,1]$.

Now we define an operator $\Phi\colon K \to K$ by

$$W(t) = (\Phi u)(t) := \begin{cases} B_0 \circ G\left(\int_0^\sigma a(r) f(u(r))dr\right) + \int_0^t G\left(\int_s^\sigma a(r) f(u(r))dr\right) ds, \\ \quad 0 \le t \le \sigma, \\ B_1 \circ G\left(\int_\sigma^1 a(r) f(u(r))dr\right) + \int_t^1 G\left(\int_\sigma^s a(r) f(u(r))dr\right) ds, \\ \quad \sigma \le t \le 1, \end{cases}$$

for each $u \in K$, where $\sigma = 0$ if $W'(0) = 0$; $\sigma = 1$ if $W'(1) = 0$; otherwise, σ is a solution of the equation

$$
\begin{aligned}
z_0(x) =& \ z_1(x), \\
z_0(x) :=& \ B_0 \circ G\left(\int_0^x a(r)f(u(r))dr\right) \\
&+ \int_0^x G\left(\int_s^x a(r)f(u(r))dr\right) ds, \qquad 0 \le x < 1, \\
z_1(x) :=& \ B_1 \circ G\left(\int_x^1 a(r)f(u(r))dr\right) \\
&+ \int_x^1 G\left(\int_x^s a(r)f(u(r))dr\right) ds, \qquad 0 < x \le 1.
\end{aligned}
\tag{2.4}
$$

The equation (2.4) has at least one solution in $(0,1)$, because $z_0(x)$ is a nondecreasing continuous function defined on $[0,1)$ with $z_0(0) = 0$ and $z_1(x)$ a nonincreasing continuous function defined on $(0,1]$ with $z_1(1) = 0$. Moreover, if $\sigma_1, \sigma_2 \in [0,1]$, $\sigma_1 < \sigma_2$, are solutions of (2.4), then we have $a(r)f(u(r)) \equiv 0$ on $[\sigma_1, \sigma_2]$. Therefore, the operator Φ is well defined.

From the definition of Φ, we deduce that for each $u \in K$, $W = \Phi u \in K$ and satisfies (1.4) and $W(\sigma)$ is the maximum value of W on $[0,1]$, since

$$
W'(t) = \begin{cases} G\left(\int_t^\sigma a(r)f(u(r))dr\right) \ge 0, & 0 < t \le \sigma, \\ -G\left(\int_\sigma^t a(r)f(u(r))dr\right) \le 0, & \sigma \le t < 1, \end{cases}
$$

is continuous and nonincreasing in $(0,1)$ and $W'(\sigma) = 0$. Moreover,

$$
[g(W'(t))]' = -a(t)f(u(t)) \quad \text{a.e. in } (0,1).
$$

This shows that $\Phi(K) \subset K$ and each fixed point of Φ in K is a solution of (1.3)–(1.4).

Lemma 2. *Assume that $f(u)$ is continuous on $[0,+\infty)$. Then (1.3)–(1.4) has a positive solution $u \in K$. Moreover, there exist two positive numbers R_1, R_2 such that*

$$
0 < R_1 \le \|u\| \le R_2.
$$

Proof. Let $u \in K$ and $W = \Phi u$. The proof is similar to that of Theorem 1. Under our hypotheses, we need to distinguish the following eight cases:

(1) $W(0) - B_0(W'(0)) = 0$, $W(1) + B_1(W'(1)) = 0$, $B_0(v) \not\equiv 0$, $B_1(v) \not\equiv 0$;
(2) $W(0) = 0$, $W(1) + B_1(W'(1)) = 0$, $B_1(v) \not\equiv 0$;
(3) $W(0) - B_0(W'(0)) = 0$, $W(1) = 0$, $B_0(v) \not\equiv 0$;
(4) $W(0) = 0$, $W(1) = 0$;
(5) $W(0) - B_0(W'(0)) = 0$, $W'(1) = 0$, $B_0(v) \not\equiv 0$;
(6) $W(0) = 0$, $W'(1) = 0$;
(7) $W'(0) = 0$, $W(1) + B_1(W'(1)) = 0$, $B_1(v) \not\equiv 0$; and
(8) $W'(0) = 0$, $W(1) = 0$.

First we deal with case (1). In this case, it is easy to see that $\Phi\colon K \to K$ is completely continuous.

Now suppose that $f_0 = 0$ and $f_\infty = +\infty$. Without loss of generality, we may assume that

$$
0 \le B_0(v) \le bv \quad \text{for all } v \ge 0.
$$

Since $f_0 = 0$, we choose $R_1 > 0$ so that

$$0 \le f(u) \le (\varepsilon u)^{p-1} \quad \text{whenever } 0 \le u \le R_1, \tag{2.5}$$

where $\varepsilon > 0$ satisfies

$$\varepsilon(b+1)G\left(\int_0^1 a(r)dr\right) < 1. \tag{2.6}$$

Thus, if $u \in K$ and $\|u\| = R_1$, then from (2.5) and (2.6)

$$\begin{aligned} \|W\| = W(\sigma) \le\ & B_0 \circ G\left(\int_0^1 a(r)f(u(r))dr\right) + G\left(\int_0^1 a(r)f(u(r))dr\right) \\ \le\ & B_0\left(\varepsilon R_1 G\left(\int_0^1 a(r)dr\right)\right) + \varepsilon R_1 G\left(\int_0^1 a(r)dr\right) \\ \le\ & R_1\varepsilon(b+1)G\left(\int_0^1 a(r)dr\right) < R_1 = \|u\|. \end{aligned} \tag{2.7}$$

Now, if we let $\Omega_1 := \{u \in C[0,1]; \|u\| < R_1\}$, then (2.7) shows that

$$\|\Phi u\| < \|u\| \quad \forall u \in K \cap \partial\Omega_1. \tag{2.8}$$

Further, since $f_\infty = +\infty$, there exists $R_2 > R_1/\delta$ such that

$$f(u) \ge (Mu)^{p-1} \quad \text{whenever } u \ge \delta R_2, \tag{2.9}$$

where $M > 0$ is chosen so that

$$\delta LM > 2, \qquad L := \min\{y(x); \delta \le x \le 1-\delta\}. \tag{2.10}$$

Let $\Omega_2 = \{u \in C[0,1]; \|u\| < R_2\}$. Then for $u \in K$ and $\|u\| = R_2$, from (2.3) and (2.10), we have

$$\begin{aligned} 2\|W\| \ge\ & \int_0^\sigma G\left(\int_s^\sigma a(r)f(u(r))dr\right)ds + \int_\sigma^1 G\left(\int_\sigma^s a(r)f(u(r))dr\right)ds \\ \ge\ & \int_\delta^\sigma G\left(\int_s^\sigma a(r)f(u(r))dr\right)ds + \int_\sigma^{1-\delta} G\left(\int_\sigma^s a(r)f(u(r))dr\right)ds \\ \ge\ & \delta M R_2\left[\int_\delta^\sigma G\left(\int_s^\sigma a(r)dr\right)ds + \int_\sigma^{1-\delta} G\left(\int_\sigma^s a(r)dr\right)ds\right] \\ \ge\ & \delta LMR_2 > 2R_2 = 2\|u\|, \quad \text{if } \sigma \in [\delta, 1-\delta], \\ \|W\| \ge\ & \int_\delta^{1-\delta} G\left(\int_\delta^s a(r)f(u(r))dr\right)ds \\ \ge\ & \delta LMR_2 > 2R_2 > \|u\|, \quad \text{if } \sigma > 1-\delta, \\ \|W\| \ge\ & \int_\delta^{1-\delta} G\left(\int_\delta^s a(r)f(u(r))dr\right)ds \\ \ge\ & \delta LMR_2 > \|u\|, \quad \text{if } \sigma < \delta, \end{aligned}$$

i.e.,

$$\|\Phi u\| > \|u\| \quad \forall u \in K \cap \partial\Omega_2.$$

Therefore, by the first part of the Fixed Point Theorem, it follows that Φ has a fixed point $u \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. Furthermore, since $0 < R_1 < \|u\| < R_2$, it follows from (2.2) that $u(t) > 0$ for $t \in (0,1)$. This shows that the fixed point u is a positive solution of (1.3)–(1.4).

Suppose next that $f_0 = +\infty$ and $f_\infty = 0$. Since $f_0 = +\infty$, we may choose $R_1 > 0$ such that

$$f(u) \geq (Mu)^{p-1} \quad \text{whenever } 0 \leq u \leq R_1, \tag{2.11}$$

where $M > 0$ satisfies (2.10). Then for $u \in K$ and $\|u\| = R_1$, we have

$$\begin{aligned}
2\|W\| \geq &\int_\delta^\sigma G\left(\int_s^\sigma a(r)f(u(r))dr\right)ds + \int_\sigma^{1-\delta} G\left(\int_\sigma^s a(r)f(u(r))dr\right)ds \\
\geq &\ \delta M R_1 \left[\int_\delta^\sigma G\left(\int_s^\sigma a(r)dr\right)ds + \int_\sigma^{1-\delta} G\left(\int_\sigma^s a(r)dr\right)ds\right] \\
\geq &\ \delta L M R_1 > 2R_1 = 2\|u\|, \quad \text{if } \sigma \in [\delta, 1-\delta], \\
\|W\| \geq &\int_\delta^{1-\delta} G\left(\int_s^{1-\delta} a(r)f(u(r))dr\right)ds \\
\geq &\ \delta L M R_1 > 2R_1 > \|u\|, \quad \text{if } \sigma > 1-\delta, \\
\|W\| \geq &\int_\delta^{1-\delta} G\left(\int_\delta^s a(r)f(u(r))dr\right)ds \\
\geq &\ \delta L M R_1 > \|u\|, \quad \text{if } \sigma < \delta,
\end{aligned}$$

i.e.,

$$\|\Phi u\| > \|u\| \quad \forall u \in K \cap \partial\Omega_1.$$

Now, since $f_\infty = 0$, there exists $R_0 > 0$ so that

$$f(u) \leq (\varepsilon u)^{p-1} \quad \text{whenever } u \geq R_0,$$

where $\varepsilon > 0$ satisfies (2.6).

If f is unbounded, then we choose $R_2 > R_0 + R_1$ so that

$$f(u) \leq f(R_2) \quad \text{for } 0 \leq u \leq R_2.$$

(We are able to do this since f is unbounded.) For $u \in K$ and $\|u\| = R_2$, we have

$$\begin{aligned}
\|W\| \leq &\ B_0 \circ G\left(\int_0^1 a(r)f(u(r))dr\right) + G\left(\int_0^1 a(r)f(u(r))dr\right) \\
\leq &\ B_0 \circ G(f(R_2))G\left(\int_0^1 a(r)dr\right) + G(f(R_2))G\left(\int_0^1 a(r)dr\right) \\
\leq &\ (b+1)\varepsilon R_2 G\left(\int_0^1 a(r)dr\right) < R_2 = \|u\|.
\end{aligned}$$

If f is bounded, say $f(u) \le N^{p-1}$ for all $u \ge 0$. In this case, let $R_2 > R_1 + N/\varepsilon$. Then for $u \in K$ with $\|u\| = R_2$, we have

$$\begin{aligned} \|W\| \le\ & B_0 \circ G\left(\int_0^1 a(r)f(u(r))dr\right) + G\left(\int_0^1 a(r)f(u(r))dr\right) \\ \le\ & B_0 \circ \left(NG\left(\int_0^1 a(r)dr\right) + N\left(\int_0^1 a(r)dr\right)\right) \\ \le\ & (b+1)NG\left(\int_0^1 a(r)dr\right) \\ <\ & (b+1)\varepsilon R_2 G\left(\int_0^1 a(r)dr\right) < R_2 = \|u\|. \end{aligned}$$

Therefore, in either case we may put $\Omega_2 := \{u \in C[0,1]; \|u\| < R_2\}$, and we have

$$\|\Phi u\| < \|u\| \quad \forall u \in K \cap \partial\Omega_2.$$

By the second part of Theorem 2, it follows that (1.3)–(1.4) has a positive solution.

We can deal with the remaining possibilities in a similar way and therefore we conclude that Lemma 2 is valid.

Lemma 3. *Assume that $f(u)$ is discontinuous on $[0,+\infty)$. Then the conclusion of Lemma 2 still holds.*

Proof. Applying linear approximation in neighborhoods of discontinuity points of f, we can construct a sequence $\{f_j(u)\}_{j=1}^{\infty}$ satisfying the following conditions:

(i) $f_j \in C[0,+\infty)$ and $0 \le f_j(u) \le f_{j+1}(u)$ on $[0,+\infty)$, $j = 1, 2, \ldots$,
(ii) $\lim_{j\to\infty} f_j(u) = f(u)$ pointwise on $[0,+\infty)$,

by (H2). Lemma 2 tells us that the boundary value problem (1.3)–(1.4) with $f = f_j$ has a positive solution

(2.12)
$$u_j(t) = \begin{cases} B_0 \circ G\left(\int_0^{\sigma_j} a(r)f_j(u_j(r))dr\right) + \int_0^t G\left(\int_s^{\sigma_j} a(r)f_j(u_j(r))dr\right)ds, \\ \qquad 0 \le t \le \sigma_j, \\ B_1 \circ G\left(\int_{\sigma_j}^1 a(r)f_j(u_j(r))dr\right) + \int_t^1 G\left(\int_{\sigma_j}^s a(r)f_j(u_j(r))dr\right)ds, \\ \qquad \sigma_j \le t \le 1, \end{cases}$$

where σ_j is a maximum point of $u_j(t)$. Moreover, $0 < R_1 \le \|u_j\| \le R_2$ with R_1, R_2 independent of j. From

$$\int_0^1 |u_j'(t)|dt \le 2\|u_j\| \le 2R_2, \qquad j = 1, 2, \ldots,$$

it follows that $\{u_j(t)\}_{j=1}^{\infty}$ is equicontinuous on $[0,1]$. Thus the existence of uniformly convergent subsequences follows from the Arzela-Ascoli Theorem. We may without loss of generality assume that $\{u_j(t)\}_{j=1}^{\infty}$ converges to $u(t)$ uniformly on $[0,1]$, $R_1 \le \|u\| \le R_2$, and $\{\sigma_j\}_{j=1}^{\infty}$ is a monotone (nondecreasing or nonincreasing) sequence which tends to $\sigma \in [0,1]$. We apply Fatou's Lemma to obtain

(2.13)

$$u(t) \geq \begin{cases} B_0 \circ G\left(\int_0^\sigma a(r) \varliminf_{j\to\infty} f(u_j(r))dr\right) + \int_0^t G\left(\int_s^\sigma a(r) \varliminf_{j\to\infty} f(u_j(r))dr\right) ds, \\ \qquad 0 \leq t \leq \sigma, \\ B_1 \circ G\left(\int_\sigma^1 a(r) \varliminf_{j\to\infty} f(u_j(r))dr\right) + \int_t^1 G\left(\int_\sigma^s a(r) \varliminf_{j\to\infty} f(u_j(r))dr\right) ds, \\ \qquad \sigma \leq t \leq 1. \end{cases}$$

On the other hand, we have

$$u(t) \leq \begin{cases} B_0 \circ G\left(\int_0^{\sigma_j} a(r)f(u_j(r))dr\right) + \int_0^t G\left(\int_s^{\sigma_j} a(r)f(u_j(r))dr\right) ds, \\ \qquad 0 \leq t \leq \sigma_j, \\ B_1 \circ G\left(\int_{\sigma_j}^1 a(r)f(u_j(r))dr\right) + \int_t^1 G\left(\int_{\sigma_j}^s a(r)f(u_j(r))dr\right) ds, \\ \qquad \sigma_j \leq t \leq 1. \end{cases}$$

Letting $j \to \infty$ in the above, we get

$$u(t) \leq \begin{cases} B_0 \circ G\left(\int_0^{\sigma} a(r)f(u(r))dr\right) + \int_0^t G\left(\int_s^{\sigma} a(r)f(u(r))dr\right) ds, \\ \qquad 0 \leq t \leq \sigma, \\ B_1 \circ G\left(\int_{\sigma}^1 a(r)f(u(r))dr\right) + \int_t^1 G\left(\int_{\sigma}^s a(r)f(u(r))dr\right) ds, \\ \qquad \sigma \leq t \leq 1. \end{cases} \tag{2.14}$$

The inequality in (2.13) and (2.14) becomes an equality, since

$$\varliminf_{j\to\infty} f(u_j(r)) \geq f(u(r)) \quad \text{for all } r \in [0,1].$$

Therefore, the function $u(t)$ is a positive solution of (1.3)–(1.4). This completes the proof of the lemma.

Theorem 3 is proved.

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