

ON KORENBLUM'S MAXIMUM PRINCIPLE

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(Communicated by Theodore W. Gamelin)

ABSTRACT. If f and g are analytic functions in the unit disk and $\|\cdot\|$ is the Bergman norm, conditions are studied under which there exists an absolute constant c such that $|f(z)| \geq |g(z)|$ for $c \leq |z| < 1$ implies $\|f\| \geq \|g\|$.

In [1] Korenblum proved the following theorem:

Consider $f, g \in A_2(\mathbb{D})$ such that each zero of g is a zero of f (according to the multiplicity) and $|f(z)| \geq |g(z)|$ for $e^{-2}/2 \leq |z| < 1$. Then

$$\int_{\mathbb{D}} |f(z)|^2 dm_2 \geq \int_{\mathbb{D}} |g(z)|^2 dm_2.$$

In the same paper and in [3] he stated the conjecture:

There exists an absolute constant c with $0 < c < 1$ such that for functions $f, g \in A_2(\mathbb{D})$

$$(*) \quad (|f(z)| \geq |g(z)| \text{ for } c \leq |z| < 1) \Rightarrow \int_{\mathbb{D}} |f(z)|^2 dm_2 \geq \int_{\mathbb{D}} |g(z)|^2 dm_2.$$

That means that the conjecture could be proved for $c = e^{-2}/2$ under an additional assumption on the zeroes of f and g . In this paper other conditions are formulated such that $(*)$ holds and the conjecture is reduced to a weaker one. We make use of the following notation: for a meromorphic function h in \mathbb{D} define $\mathcal{N}(r, h)$ as the set of poles of h in $\{|z| < r\}$ and $n(r, h)$ as the number of poles of h in $\{|z| < r\}$ (according to the multiplicity). We prove:

Theorem 1. For $0 < \epsilon < \frac{1}{2}$ a constant $c_\epsilon > 0$ exists such that for any pair $f, g \in A_2(\mathbb{D})$

$$\begin{aligned} & \left(|f(z)| \geq |g(z)| \text{ for } c_\epsilon \leq |z| < 1 \text{ and } \mathcal{N}\left(1 - \epsilon, \frac{f}{g}\right) \setminus \mathcal{N}\left(\epsilon, \frac{f}{g}\right) \neq \emptyset \right) \\ & \Rightarrow \int_{\mathbb{D}} |f(z)|^2 dm_2 \geq \int_{\mathbb{D}} |g(z)|^2 dm_2. \end{aligned}$$

Received by the editors February 16, 1994 and, in revised form, December 1, 1994.
1991 *Mathematics Subject Classification.* Primary 30C80, 30H05.

Theorem 2. For $0 < \epsilon < \frac{1}{2}$ a constant $c_\epsilon > 0$ exists such that for any pair $f, g \in A_2(\mathbb{D})$

$$\begin{aligned} & \left(|f(z)| \geq |g(z)| \text{ for } c_\epsilon \leq |z| < 1 \text{ and } n\left(1 - \epsilon, \frac{1}{f}\right) \neq n\left(1 - \epsilon, \frac{1}{g}\right) \right) \\ & \Rightarrow \int_{\mathbb{D}} |f(z)|^2 dm_2 \geq \int_{\mathbb{D}} |g(z)|^2 dm_2. \end{aligned}$$

This means that the conjecture holds if we consider functions f and g such that g has at least one zero in the annulus $\{\epsilon_0 < |z| < 1 - \epsilon_0\}$ which is not a zero of f according to the multiplicity (Theorem 1), or if the number of zeroes of f and g in a circle $\{|z| < 1 - \epsilon_0\}$ does not coincide (Theorem 2).

Theorem 3. There exists a constant $c > 0$ such that for any pair $f, g \in A_2(\mathbb{D})$

$$\begin{aligned} & \left(|f(z)| \geq |g(z)| \text{ for } c \leq |z| < 1 \text{ and } n\left(1, \frac{g}{f}\right) = n\left(1, \frac{f}{g}\right) = 1 \right) \\ & \Rightarrow \int_{\mathbb{D}} |f(z)|^2 dm_2 \geq \int_{\mathbb{D}} |g(z)|^2 dm_2. \end{aligned}$$

Theorem 4. For $\alpha \in \mathbb{D}$ let $A_2^\alpha(\mathbb{D}) = \{h \in A_2(\mathbb{D}) | h(\alpha) = 0\}$. If $0 < r_1 < \frac{1}{2}$ and $|a| < \frac{r_1}{4}$, then

$$\begin{aligned} T_a : A_2^0(\mathbb{D}) & \rightarrow A_2^\alpha(\mathbb{D}) \\ h & \mapsto (T_a h)(z) := \frac{h(z)}{z} \cdot \frac{z - a}{1 - \bar{a}z} \cdot \frac{1 + |a|r_1}{1 + \frac{|a|}{r_1}} \end{aligned}$$

defines a contractive operator.

Proof of Theorem 1. Let $0 < \epsilon < \frac{1}{2}$ be given, and assume that $f_n, g_n \in A_2(\mathbb{D})$ exist with

$$|f_n(z)| \geq |g_n(z)| \quad \text{for } c_n \leq |z| < 1 \quad (c_n \searrow 0)$$

and

$$\int_{\mathbb{D}} |f(z)|^2 dm_2 < \int_{\mathbb{D}} |g_n(z)|^2 dm_2.$$

Define

$$\varphi_n(z) = \frac{g_n(z)}{f_n(z)}, \quad z \in \mathbb{D}.$$

Then φ_n is meromorphic in \mathbb{D} and $|\varphi_n(z)| \leq 1$ for $c_n \leq |z| < 1$. Therefore $(\varphi_n)_{n \in \mathbb{N}}$ is normal in $\{0 < |z| < 1\}$ and w.l.o.g. we may assume that φ_n tends to a limit function φ locally uniformly in $\{0 < |z| < 1\}$. φ is analytic in $0 < |z| < 1$ and the singularity in 0 is removable. Hence we have $|\varphi(z)| \leq 1$ in $\{|z| < 1\}$. Then the condition

$$\mathcal{N}\left(1 - \epsilon, \frac{f_n}{g_n}\right) \setminus \mathcal{N}\left(\epsilon, \frac{f_n}{g_n}\right) \neq \emptyset$$

implies that each φ_n has at least one zero in $\epsilon \leq |z| < 1 - \epsilon$. Hence we have $|\varphi(z)| < 1$ in $|z| < 1$. If $R := \{\frac{1}{2} \leq |z| \leq \frac{3}{4}\}$, $\eta_0 > 0$ exists such that $|\varphi(z)| \leq 1 - 2\eta_0$ for $z \in R$. Then $n_0 \in \mathbb{N}$ exists with $n_0 \cdot \eta_0 \geq 1$. Choose $n_1 \geq n_0$ with $c_n \leq \frac{1}{n_0}$ and

$$\left| \frac{g_n(z)}{f_n(z)} \right| = |\varphi_n(z)| \leq \sqrt{1 - \eta_0} \quad \text{for } z \in R,$$

if $n \geq n_1$. For $h \in A_2(\mathbb{D})$ with $h(z) = \sum_{k=0}^\infty a_k z^k$ the function

$$J(r, h) := \int_{|z|<r} |h(z)|^2 dm_2 = \pi \cdot \sum_{k=0}^\infty \frac{|a_k|^2}{k+1} r^{2k+2} \quad (0 \leq r < 1)$$

satisfies the inequality

$$J(r + \rho, h) - J(r, h) \leq J(R + \rho, h) - J(R, h)$$

for $0 \leq r < R < 1$ and $R + \rho < 1$, since the same estimate holds for the powers r^{2k+2} . For $n \geq n_1$ we get

$$\begin{aligned} \int_R |f_n(z)|^2 - |g_n(z)|^2 dm_2 &\geq \int_R \frac{1}{1-\eta_0} |g_n(z)|^2 - |g_n(z)|^2 dm_2 \\ &> \int_R \eta_0 |g_n(z)|^2 dm_2 = \eta_0 \cdot \sum_{j=0}^{n_0-1} \int_{\frac{1}{2} + \frac{j}{n_0} < |z| < \frac{1}{2} + \frac{j+1}{n_0}} |g_n(z)|^2 dm_2 \\ &\geq n_0 \cdot \eta_0 \cdot \int_{\frac{1}{2} < |z| < \frac{1}{2} + \frac{1}{n_0}} |g_n(z)|^2 dm_2 \geq \int_{|z| < c_n} |g_n(z)|^2 dm_2 \end{aligned}$$

and therefore

$$\int_{\mathbb{D}} |f_n(z)|^2 - |g_n(z)|^2 dm_2 > 0.$$

This is a contradiction and proves Theorem 1. □

Proof of Theorem 2. We start with sequences f_n, g_n as in the proof of Theorem 1. Then the result follows as above, if we are able to show that $|\varphi(z)| < 1$ for $|z| < 1$ holds for the limit function φ .

Suppose $|\varphi(z)| \equiv 1$. It follows that for $n \geq n_0$ all zeroes and poles of φ_n lie in $\{|z| < \epsilon\} \cup \{1 - \epsilon < |z| < 1\}$. We also have $n(\epsilon, \varphi_n) = n(\epsilon, \frac{1}{\varphi_n})$ for all, but finitely many $n \in \mathbb{N}$. To see this let $a_\mu^{(n)}$ ($\mu = 1, \dots, n(\epsilon, \frac{1}{\varphi_n})$) be the zeroes and $b_\nu^{(n)}$ ($\nu = 1, \dots, n(\epsilon, \varphi_n)$) be the poles of φ_n in $|z| < \epsilon$ ($n \geq n_0$). W.l.o.g. we have $\varphi_n(0) \neq 0, \infty$ ($n \in \mathbb{N}$). Otherwise choose $|z_0| < \frac{\epsilon}{2}$ with $\varphi_n(z_0) \neq 0, \infty$ for $n \in \mathbb{N}$ and prove $n(\frac{\epsilon}{2}, \tilde{\varphi}_n) = n(\frac{\epsilon}{2}, \frac{1}{\tilde{\varphi}_n})$ for $\tilde{\varphi}_n(z) = \varphi_n(z + z_0)$. The Poisson-Jensen formula implies

$$\log |\varphi_n(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |\varphi_n(r_j e^{it})| dt - \sum_{\mu=1}^{n(\epsilon, \frac{1}{\varphi_n})} \log \frac{r_j}{|a_\mu^{(n)}|} + \sum_{\nu=1}^{n(\epsilon, \varphi_n)} \log \frac{r_j}{|b_\nu^{(n)}|}$$

for $\epsilon < r_1 < r_2 < 1 - \epsilon$. Then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |\varphi_n(r_2 e^{it})| - \log |\varphi_n(r_1 e^{it})| dt = \left[n\left(\epsilon, \frac{1}{\varphi_n}\right) - n(\epsilon, \varphi_n) \right] \cdot \log \frac{r_2}{r_1}$$

and the assertion follows, since the left-hand side tends to 0. The proposition

$$n\left(1 - \epsilon, \frac{1}{f_n}\right) \neq n\left(1 - \epsilon, \frac{1}{g_n}\right)$$

implies that the number of zeroes of f_n and g_n in $\epsilon \leq |z| < 1 - \epsilon$ does not coincide, if n is sufficiently large. Then $|\varphi(z)| < 1$ follows as in Theorem 1. This is a contradiction and proves Theorem 2. □

If we want to prove the existence of an absolute constant c such that $(*)$ holds under Korenblum’s additional assumption on the zeroes of f and g (compare [1]), it is a consequence of the classical maximum principle that we only have to study the case that f/g has at least one zero in $|z| < c$. Then the assertion follows from Theorem 2.

In the following we want to reduce conjecture $(*)$. We suppose that it does not hold and consider $f_n, g_n \in A_2(\mathbb{D})$ with

$$|f_n(z)| \geq |g_n(z)| \quad \text{for } c_n \leq |z| < 1 \quad (c_n \searrow 0)$$

and

$$\int_{\mathbb{D}} |f_n(z)|^2 dm_2 < \int_{\mathbb{D}} |g_n(z)|^2 dm_2.$$

If φ_n and φ are defined as above, the proof of Theorem 1 implies $|\varphi(z)| \equiv 1$. W.l.o.g. we may assume that f_n and g_n are bounded and that all zeroes and poles of φ_n lie in $\{|z| < \frac{1}{4}\}$ or even in a smaller circle. Otherwise replace f_n and g_n by

$$\tilde{f}_n(z) := f_n(z/2), \quad \tilde{g}_n(z) := g_n(z/2).$$

It is a consequence of Theorem 2 that $n(\frac{1}{4}, \varphi_n) = n(\frac{1}{4}, \frac{1}{\varphi_n})$ for n sufficiently large. Define $a_\mu^{(n)}, b_\nu^{(n)}$ as above. If we consider

$$f_n \left(\frac{a_1^{(n)} - b_1^{(n)}}{|a_1^{(n)} - b_1^{(n)}|} \cdot z + b_1^{(n)} \right) \quad \text{and} \quad g_n \left(\frac{a_1^{(n)} - b_1^{(n)}}{|a_1^{(n)} - b_1^{(n)}|} \cdot z + b_1^{(n)} \right)$$

instead of f_n and g_n , we can assume that each g_n has a positive zero and that 0 is a zero of each f_n . For $m(n) = n(\frac{1}{4}, \varphi_n)$ and

$$A_n(z) = \prod_{\mu=1}^{m(n)} \frac{z - a_\mu^{(n)}}{1 - \bar{a}_\mu^{(n)}z}, \quad B_n(z) = \prod_{\nu=1}^{m(n)} \frac{z - b_\nu^{(n)}}{1 - \bar{b}_\nu^{(n)}z}$$

we have the representations

$$g_n(z) = h_n(z) \cdot A_n(z), \quad f_n(z) = h_n(z) \cdot B_n(z) \cdot e^{\psi_n(z)},$$

where h_n and ψ_n are analytic in \mathbb{D} . W.l.o.g. we may suppose $|a_\mu^{(n)}| < c_n, |b_\nu^{(n)}| < c_n$ for $\mu, \nu = 1, \dots, m(n)$. The proposition gives

$$\left| \frac{B_n(z)}{A_n(z)} \right| \cdot e^{\operatorname{Re} \psi_n(z)} \geq 1$$

for $c_n \leq |z| < 1$ and the maximum principle implies $\operatorname{Re} \psi_n(z) > 0$ in $|z| < 1$, since $\left| \frac{B_n(z)}{A_n(z)} \right| \rightarrow 1$ for $|z| \rightarrow 1$. Then we have (compare [4], p. 140)

$$\operatorname{Re} \psi_n(0) \cdot \frac{1 + |z|}{1 - |z|} \geq \operatorname{Re} \psi_n(z) \geq \operatorname{Re} \psi_n(0) \cdot \frac{1 - |z|}{1 + |z|}$$

for $|z| < 1$ and therefore

$$\operatorname{Re} \psi_n(0) \geq \frac{1 - c_n}{1 + c_n} \cdot \log \max_{|\zeta|=c_n} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|.$$

This implies the estimate

$$\operatorname{Re} \psi_n(z) \geq \frac{1 - c_n}{1 + c_n} \cdot \log \max_{|\zeta|=c_n} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right| \cdot \frac{1 - |z|}{1 + |z|}$$

for $|z| < 1$. The method in the proof of Theorem 2 to show that the numbers of zeroes and poles of φ_n coincide applied to the quotient A_n/B_n with $r_1 = r < 1$ and $r_2 = 1$ gives

$$\frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{A_n}{B_n}(re^{it}) \right| dt = 0$$

and therefore

$$\max_{|\zeta|=r} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right| > 1$$

for $c_n \leq r < 1$. Hence

$$\begin{aligned} |f_n(z)| &= |h_n(z)| \cdot |B_n(z)| \cdot e^{\operatorname{Re} \psi_n(z)} \\ &\geq |h_n(z)| \cdot |B_n(z)| \cdot \max_{|\zeta|=c_n} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|^{\frac{1-c_n}{1+c_n} \cdot \frac{1-|z|}{1+|z|}} \\ &\geq |h_n(z)| \cdot |B_n(z)| \cdot \max_{|\zeta|=c_n} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|^{\frac{1}{3} \cdot \frac{1-c_n}{1+c_n}} \end{aligned}$$

for $|z| \leq \frac{1}{2}$. If $0 < r_0 < \frac{1}{2}$ is a fixed radius and $c_n < r_0$ Hadamard's three circle theorem implies

$$\max_{|\zeta|=c_n} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right| \geq \max_{|\zeta|=r_0} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|^{\frac{\log(1/c_n)}{\log(1/r_0)}}.$$

If n is sufficiently large, we then have

$$|f_n(z)| \geq |h_n(z)| \cdot |B_n(z)| \cdot \max_{|\zeta|=r_0} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|$$

for $|z| \leq \frac{1}{2}$ and

$$|h_n(z)| \cdot |B_n(z)| \cdot \max_{|\zeta|=r_0} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right| \geq |g_n(z)|$$

for $r_0 \leq |z| \leq \frac{1}{2}$. Define

$$\tilde{f}_n(z) = h_n\left(\frac{z}{2}\right) \cdot B_n\left(\frac{z}{2}\right) \cdot \max_{|\zeta|=r_0} \left| \frac{A_n(\frac{1}{2}\zeta)}{B_n(\frac{1}{2}\zeta)} \right| \quad \text{and} \quad \tilde{g}_n(z) = g_n\left(\frac{z}{2}\right)$$

for $|z| < 1$ and $0 < r_0 < 1$. Then $|\tilde{f}_n(z)| \geq |\tilde{g}_n(z)|$ for $r_0 \leq |z| < 1$ (n sufficiently large) and

$$\begin{aligned} \int_{\mathbb{D}} |\tilde{f}_n(z)|^2 dm_2 &= \int_{\mathbb{D}} |h_n(\frac{z}{2})B_n(\frac{z}{2})|^2 \cdot \max_{|\zeta|=r_0} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|^2 dm_2 \\ &= 4 \int_{|z|<\frac{1}{2}} |h_n(z)B_n(z)|^2 \cdot \max_{|\zeta|=r_0} \left| \frac{A_n(\zeta)}{B_n(\zeta)} \right|^2 dm_2 \\ &\leq 4 \int_{|z|<\frac{1}{2}} |f_n(z)|^2 dm_2 < 4 \int_{|z|<\frac{1}{2}} |g_n(z)|^2 dm_2 = \int_{\mathbb{D}} |\tilde{g}_n(z)|^2 dm_2. \end{aligned}$$

Therefore (*) follows, if we are able to prove the following conjecture:

There exists an absolute constant $0 < r_0 < \frac{1}{2}$ such that the following holds: Let $h(z)$ be a bounded analytic function in \mathbb{D} and $A(z)$ and $B(z)$ be finite Blaschke

products. If $A(z)$ and $B(z)$ have the same number of zeroes in \mathbb{D} and if each zero of $A(z)$ or $B(z)$ lies in $\{|z| < r_0\}$, then

$$(**) \quad \int_{|z| < \frac{1}{2}} |h(z) \cdot B(z)|^2 \cdot \max_{|\zeta|=r_0} \left| \frac{A(\zeta)}{B(\zeta)} \right|^2 dm_2 \geq \int_{|z| < \frac{1}{2}} |h(z) \cdot A(z)|^2 dm_2.$$

The proof of Theorem 2 yields that the zeroes of the Blaschke products may even be assumed to be contained in a smaller circle.

Proof of Theorem 3. The above considerations starting with sequences f_n and g_n such that (*) does not hold and ending with conjecture (**) show that Theorem 3 follows, if we are able to solve (**) in the case $A(z) = \frac{z-a}{1-\bar{a}z}$ ($a > 0$) and $B(z) = z$. Then we have to show the existence of $0 < r_0 < \frac{1}{2}$ such that

$$\int_{|z| < \frac{1}{2}} |h(z)|^2 |z|^2 \cdot \max_{|\zeta|=r_0} \left| \frac{\zeta - a}{(1 - \bar{a}\zeta)\zeta} \right|^2 dm_2 \geq \int_{|z| < \frac{1}{2}} |h(z)|^2 \left| \frac{z - a}{1 - \bar{a}z} \right|^2 dm_2$$

for all h analytic in \mathbb{D} , respectively

$$\begin{aligned} & \int_{|z| < \frac{1}{2}} |h(z)|^2 |z|^2 \left| \frac{1 - \bar{a}z}{1 - 4\bar{a}z} \right|^2 \cdot \max_{|\zeta|=r_0} \left| \frac{\zeta - a}{(1 - \bar{a}\zeta)\zeta} \right|^2 dm_2 \\ & \geq \int_{|z| < \frac{1}{2}} |h(z)|^2 \left| \frac{z - a}{1 - 4\bar{a}z} \right|^2 dm_2. \end{aligned}$$

For $|z| \leq \frac{1}{2}$ and $0 < r_0 < \frac{1}{8}$ we get

$$\begin{aligned} & \left| \frac{1 - \bar{a}z}{1 - 4\bar{a}z} \right| \cdot \max_{|\zeta|=r_0} \left| \frac{\zeta - a}{(1 - \bar{a}\zeta)\zeta} \right| \geq \frac{\max_{|\zeta|=r_0} \left| \frac{\zeta - a}{(1 - \bar{a}\zeta)\zeta} \right|}{\max_{|\zeta|=1/2} \left| \frac{\zeta - a}{(1 - \bar{a}\zeta)\zeta} \right|} \\ & = \frac{1 + \frac{|a|}{r_0}}{1 + |a|r_0} \cdot \frac{1 + \frac{|a|}{2}}{1 + 2|a|} \geq \frac{1 + \frac{|a|}{4r_0}}{1 + 16|a|r_0} = \max_{|\zeta|=4r_0} \left| \frac{\zeta - a}{(1 - 4\bar{a}\zeta)\zeta} \right|. \end{aligned}$$

Hence it suffices to prove the existence of $0 < r_1 < \frac{1}{2}$ such that

$$\int_{|z| < \frac{1}{2}} |h(z)|^2 |z|^2 \cdot \max_{|\zeta|=r_1} \left| \frac{\zeta - a}{(1 - \bar{a}\zeta)\zeta} \right|^2 dm_2 \geq \int_{|z| < \frac{1}{2}} |h(z)|^2 \left| \frac{z - a}{1 - 4\bar{a}z} \right|^2 dm_2$$

for all h analytic in \mathbb{D} . This inequality, transformed to the unit circle, follows from the next lemma. □

Lemma 5. *If $h \in A_2(\mathbb{D})$, $0 < r_1 \leq \frac{1}{2}$ and $0 < a < \frac{r_1}{4}$, then*

$$\int_{\mathbb{D}} |h(z)|^2 |z|^2 \cdot \max_{|\zeta|=r_1} \left| \frac{\zeta - a}{(1 - \bar{a}\zeta)\zeta} \right|^2 dm_2 \geq \int_{\mathbb{D}} |h(z)|^2 \left| \frac{z - a}{1 - \bar{a}z} \right|^2 dm_2.$$

Proof. We first show

$$(1) \quad \int_{\mathbb{D}} |z^k|^2 \cdot |z|^2 \cdot \max_{|\zeta|=r_1} \left| \frac{\zeta - a}{(1 - \bar{a}\zeta)\zeta} \right|^2 dm_2 \geq \int_{\mathbb{D}} |z^k|^2 \cdot |z|^{2 \cdot \frac{1-a}{1+a}} dm_2$$

for $0 < r_1 \leq \frac{1}{2}$ and $k \in \mathbb{N}_0$. Then (1) is equivalent to

$$\frac{\pi}{k + 2} \left(\frac{1 + \frac{a}{r_1}}{1 + a \cdot r_1} \right)^2 \geq \frac{\pi}{k + 1 + \frac{1-a}{1+a}},$$

respectively

$$\left(\frac{1}{r_1} - r_1\right) \left(2 + a \left(\frac{1}{r_1} + r_1\right)\right) \cdot (k+2) \geq \left(1 + \frac{a}{r_1}\right)^2 \frac{2}{1+a}.$$

But this is obviously true if $\frac{1}{r_1} - r_1 \geq 1$, especially for $r_1 \leq \frac{1}{2}$. Then we get

$$\int_{\mathbb{D}} |h(z)|^2 |z|^2 \max_{|\zeta|=r_1} \left| \frac{\zeta - a}{(1 - \bar{a}\zeta) \cdot \zeta} \right|^2 dm_2 \geq \int_{\mathbb{D}} |h(z)|^2 |z|^{2 \cdot \frac{1-a}{1+a}} dm_2$$

for $h \in A_2(\mathbb{D})$. If we apply [2], Lemma 4 with $\gamma = \frac{1-a}{1+a}$ we have

$$\int_{\mathbb{D}} |h(z)|^2 |z|^{2 \cdot \frac{1-a}{1+a}} dm_2 \geq \int_{\mathbb{D}} |h(z)|^2 \left| \frac{z-a}{1-\bar{a}z} \right|^2 dm_2,$$

which implies the assertion. \square

Proof of Theorem 4. Consider $h \in A_2^0(\mathbb{D})$. Then

$$\begin{aligned} \int_{\mathbb{D}} |h(z)|^2 dm_2 &= \left(\frac{1+a \cdot r_1}{1+\frac{a}{r_1}}\right)^2 \cdot \int_{\mathbb{D}} \left| \frac{h(z)}{z} \right|^2 |z|^2 \cdot \max_{|\zeta|=r_1} \left| \frac{\zeta - a}{(1 - \bar{a}\zeta)\zeta} \right|^2 dm_2 \\ &\stackrel{L.5}{\geq} \left(\frac{1+a \cdot r_1}{1+\frac{a}{r_1}}\right)^2 \cdot \int_{\mathbb{D}} \left| \frac{h(z)}{z} \right|^2 \left| \frac{z-a}{1-\bar{a}z} \right|^2 dm_2 = \int_{\mathbb{D}} |T_a(h)(z)|^2 dm_2. \quad \square \end{aligned}$$

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