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CONSTRUCTION OF INVARIANT CURVES FOR SINGULAR HOLOMORPHIC VECTOR FIELDS

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ABSTRACT. Camacho and Sad proved the existence of invariant analytic curves for germs of singular holomorphic foliations $\mathcal{F}$ over a two dimensional complex analytic variety M. Their proof is only of existential nature. Here we provide a simple constructive proof by giving criteria to choose a singular point at each blowing-up that follows an analytic invariant curve.

Our algorithm is founded on the stability by blowing-up of the property $(\star)$ introduced in the following definition.

Definition. Consider a singular holomorphic foliation $\mathcal{F}$ over a two dimensional complex analytic variety M, a normal crossings divisor E over M and a point $q \in E$. We say that the triple $(\mathcal{F}, E, q)$ has the property $(\star)$ if and only if one of the following properties holds:

$(\star)$-1: The point q lies exactly in one irreducible component S of E, which is invariant for $\mathcal{F}$ and the index $i_q(\mathcal{F}, S) \notin \mathbb{Q}_{(\geq 0)} = \{r \in \mathbb{Q}; r \geq 0\}$.

$(\star)$-2: The point q lies in two irreducible components S_+ and S_- of E (call this point a "corner"), both are invariant curves and there is a real number $a > 0$ such that:

$$\begin{aligned} i_q(\mathcal{F}, S_+) &\in \mathbb{Q}_{(\leq -a)} = \{r \in \mathbb{Q}; r \leq -a\}, \\ i_q(\mathcal{F}, S_-) &\notin \mathbb{Q}_{(\geq -1/a)} = \{r \in \mathbb{Q}; r \geq -1/a\}. \end{aligned}$$

$(\star)$-3: The point q lies exactly in one irreducible component S of E, it is a nonsingular point of $\mathcal{F}$ and S is transversal to $\mathcal{F}$ at q.

(The definition and basic properties of the index can be found in [1]).

Remark. If we have the property $(\star)$-2, then q is not a simple (irreducible) singularity. If we have either $(\star)$-1 and q is a simple singularity or $(\star)$-3, then there is a nonsingular analytic invariant curve Γ through q transversal to E.

Theorem. *Assume that $(\mathcal{F}, E, q)$ satisfies either $(\star)$-1 or $(\star)$-2. Consider the blowing-up $\pi : M' \to M$ at the point q. Let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ by π. Put $D = \pi^{-1}(q)$ and $E' = \pi^{-1}(E)$. Then there is a point $q' \in D$ such that $(\mathcal{F}', E', q')$ satisfies the property $(\star)$.*

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Proof. If π is a dicritical blowing-up we immediately get a point $q' \in D$ such that $(\mathcal{F}', E', q')$ satisfies the property $(\star)$-3.

Assume that π is non dicritical and hence D is an invariant curve for $\mathcal{F}'$.

Consider first the case that $(\mathcal{F}, E, q)$ satisfies $(\star)$-1. Let S' be the strict transform of S by π and put $\{q'\} = D \cap S'$. Let $p_1, \ldots, p_s$ be the singularities of $\mathcal{F}'$ on $D \setminus \{q'\}$. Suppose that $(\mathcal{F}', E', p_i)$ does not satisfy $(\star)$-1 for any $i = 1, \ldots, s$. Then we have that

$$i_{q'}(\mathcal{F}', D) = -1 - \sum_{i=1}^{s} i_{p_i}(\mathcal{F}', D) \in \mathbb{Q}_{(\leq -1)}.$$

Since $i_{q'}(\mathcal{F}', S') = i_q(\mathcal{F}, S) - 1 \notin \mathbb{Q}_{(\geq -1)}$, then $(\mathcal{F}', E', q')$ satisfies $(\star)$-2.

Consider now the case that $(\mathcal{F}, E, q)$ satisfies $(\star)$-2. Let S'_+ and S'_- be the strict transforms by π of S_+ and S_- respectively. Let $p_1, \ldots, p_s$ be the singularities of $\mathcal{F}'$ on $D \setminus \{q_+, q_-\}$, where $q_+ = D \cap S'_+$ and $q_- = D \cap S'_-$. If $(\mathcal{F}', E', p_i)$ does not satisfy $(\star)$-1 for $i = 1, \ldots, s$, then

$$i_{q_+}(\mathcal{F}', D) + i_{q_-}(\mathcal{F}', D) = -1 - \sum_{i=1}^{s} i_{p_i}(\mathcal{F}', D) \in \mathbb{Q}_{(\leq -1)}. \tag{1}$$

Since $(\mathcal{F}, E, q)$ satisfies the property $(\star)$-2 we have that

$$\begin{aligned} i_{q_+}(\mathcal{F}', S'_+) &= i_q(\mathcal{F}, S_+) - 1 \in \mathbb{Q}_{(\leq -(a+1))}, \\ i_{q_-}(\mathcal{F}', S'_-) &= i_q(\mathcal{F}, S_-) - 1 \notin \mathbb{Q}_{(\geq -\frac{a+1}{a})}. \end{aligned}$$

If $(\mathcal{F}', E', q_+)$ does not satisfy $(\star)$-2 then $i_{q_+}(\mathcal{F}', D) \in \mathbb{Q}_{(\geq -\frac{1}{a+1})}$. By (1) $i_{q_-}(\mathcal{F}', D) \in \mathbb{Q}_{(\leq -\frac{a}{a+1})}$ and thus $(\mathcal{F}', E', q_-)$ satisfies $(\star)$-2. □

To get an analytic invariant curve Γ for $\mathcal{F}$ at q we proceed as follows. After the blowing-up with center q we take a point p_1 in the exceptional divisor E_1 with the property $(\star)$: if the blowing-up is dicritical we get $(\star)$-3 and the algorithm stops, otherwise we get $(\star)$-1 since the sum of the indices is -1. Repeat. By reduction of singularities, in a finite number of steps we get either $(\star)$-3 or an irreducible singularity satisfying $(\star)$. This gives an analytic invariant curve Γ' transversal to the divisor that projects over Γ.

References

[1] C. Camacho and P. Sad. *Invariant varieties through singularities of holomorphic vector fields*, Ann. of Math., **115**, (1982) 579-595. MR **83m:**58062

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