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INEQUIDIMENSIONALITY OF HILBERT SCHEMES

MEI-CHU CHANG

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Abstract. We give a lower bound on the number of distinct dimensions of irreducible components of the Hilbert scheme of codimension 2 subvarieties in $\mathbb{P}^n$, for $n \leq 5$ (respectively, the moduli space of surfaces or 3-folds) in terms of the Hilbert polynomial (resp. Chern numbers).

Let $Hilb_P$ be the Hilbert scheme of subvarieties in the projective space with fixed Hilbert polynomial P (respectively, let $\mathcal{M}$ be a moduli space of varieties with fixed Chern numbers). It is known that $Hilb_P$ (resp. $\mathcal{M}$) has finitely many irreducible components and that the number of these components is bounded by some function of the Hilbert polynomial (resp. the Chern numbers). For work on the number of components of the Hilbert scheme (resp. the moduli space), see [EHM] for curves in $\mathbb{P}^3$ and [Ch1] for codimension 2 subvarieties in $\mathbb{P}^n$ with $n \leq 5$ (resp. [Ca1], [Ca2], [Ca3], [M] for surfaces and [Ch1] for surfaces and 3-folds).

The next question to ask is whether the Hilbert scheme (resp. moduli space) is equidimensional if it is reducible. Catanese [Ca3] has shown that for $\mathcal{M}$, the moduli space of surfaces, the number of distinct dimensions can be arbitrarily large.

In this note we study the number of distinct dimensions of the components of the Hilbert scheme $Hilb_P$ (resp. moduli space $\mathcal{M}$) parametrizing subschemes with intersection numbers H^iK^{n-2-i} (resp. Chern numbers), where H is the hyperplane class and K is the canonical class.

We define

$$\begin{aligned} n(d, HK, K^2) &= \#\{\dim H \mid H \text{ is a component of the Hilbert scheme of} \\ &\qquad \text{surfaces in } \mathbb{P}^4 \text{ with intersection numbers } d, HK, K^2\}, \\ n(d, H^2K, HK^2, K^3) &= \#\{\dim H \mid H \text{ is a component of the Hilbert schemes of} \\ &\qquad \text{3-folds in } \mathbb{P}^5 \text{ with intersection numbers } d, H^2K, HK^2, K^3\}, \\ n(K^2, c_2) &= \#\{\dim H \mid H \text{ is a component of the moduli space of} \\ &\qquad \text{surfaces with Chern numbers } K^2, c_2\}, \\ n(K^3, c_1c_2, c_3) &= \#\{\dim H \mid H \text{ is a component of the moduli space} \\ &\qquad \text{of 3-folds with Chern numbers } K^3, c_1c_2, c_3\}. \end{aligned}$$

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Notation. Let d be a function of r. We write $d = O(r^\alpha)$ if there are positive constants c_1 and c_2 such that $c_1 r^\alpha \le d \le c_2 r^\alpha$ for r sufficiently large.

What we can show is the following

Theorem. *There is an infinite family of triples (respectively, quadruples) of integers (d, HK, K^2) (resp. (d, H^2K, HK^2, K^3)) such that the Hilbert scheme of surfaces in $\mathbb{P}^4$ (resp. 3-folds in $\mathbb{P}^5$) with intersection numbers d, HK, K^2 (resp. d, H^2K, HK^2, K^3) has irreducible components of at least $O(y^{3/4})$ (resp. $O(y)$) distinct dimensions, where $y = K^2$ (resp. $y = K^3$).*

In the statement above, replacing Hilbert scheme by moduli space and intersection numbers by Chern numbers, we have at least $O(y^{5/4})$ many distinct dimensions for surfaces and $O(y)$ for 3-folds, y being the self intersection number of the canonical class K. More precisely,

(i) *For $K^2 \gg 0$ there exist d and HK such that $n(d, HK, K^2) \ge O((K^2)^{\frac{3}{4}})$. Here $d = O((K^2)^{\frac{1}{2}})$ and $HK = O((K^2)^{\frac{3}{4}})$.*
(ii) *For $K^3 \gg 0$ there exist $d = O((K^3)^{\frac{2}{5}}), H^2K = O((K^3)^{\frac{3}{5}})$ and $HK^2 = O((K^3)^{\frac{4}{5}})$ such that $n(d, H^2K, HK^2, K^3) \ge O((K^3))$.*
(iii) *For $K^2 \gg 0$ there exists $c_2 = O(K^2)$ such that $n(K^2, c_2) \ge O((K^2)^{\frac{5}{4}})$.*
(iv) *For $K^3 \gg 0$ there exist c_1c_2 and c_3 such that $n(K^3, c_1c_2, c_3) \ge O((K^3))$. Here $c_1c_2 = O((K^2))$ and $c_3 = O((K^2))$.*

Remark 1. If $n(d, g)$ is the analogous notion for curves in $\mathbb{P}^3$ of degree d and genus g, then for $g \gg 0$ there exists $d = O(g^{\frac{2}{3}})$ such that $n(d, g) \ge O(g)$.

These results are obtained by studying the irreducible components corresponding to projectively normal subvarieties of codimension two in $\mathbb{P}^n$. Counting the number of distinct dimensions of such components boils down to the following

Problem. Let $c_1 \le \cdots \le c_r$ be any increasing sequence of positive integers within a certain range. If the symmetric functions $s_j(\bar{c}) = \sum_{i=1}^{r} c_i^j$, $j = 1, \ldots, 5$, are fixed, how many values are taken on by the asymmetric, homogeneous function $t_j(\bar{c}) = \sum_{i=1}^{5} i c_i^j$ or $u(\bar{c}) = \sum_{i>j} c_i^2 c_j$?

Our approach is to study the function $f = (s_1, \ldots, s_5, t_1, \ldots, t_3, u)$ defined on the positive *real* numbers. We show that the image of f contains a "box" of maximal size. Hence the range of the fiber over a fixed $(s_1, \ldots, s_5)$ is as large as possible. Then we approximate the integral points in the fiber by the images of integral points (points with integral coordinates).

Question. Can one use the idea here and some formal deformation theory [R] to obtain a much larger number of distinct dimensions of components for the moduli spaces? (See Remark 4 at the end of §1.)

The paper is organized in the following way. In section 1 we reduce our geometric question to a combinatorial one and prove the theorem. Section 2 is devoted to the combinatorial problem stated above.

§1

Let X be a projectively normal 3-fold in $\mathbb{P}^5$ with resolution

$$0 \to \bigoplus_{1}^{r+2} \mathcal{O}(-a_i) \to \bigoplus_{-2}^{r} \mathcal{O}(-b_j) \to \mathcal{I}_X \to 0 \tag{1}$$

where $\{a_i\}$ and $\{b_j\}$ are two disjoint *increasing* sequences.

(A) (Ellingsrud [E]) Let H be a component of the Hilbert scheme containing X. Then

$$\begin{aligned} \dim H = \sum_{a_i \geq b_j} \binom{a_i - b_j + 5}{5} + \sum_{b_j \geq a_i} \binom{b_j - a_j + 5}{5} \\ - \sum_{i \geq j} \binom{a_i - a_j + 5}{5} - \sum_{i \geq j} \binom{b_i - b_j + 5}{5} + 1. \end{aligned} \tag{2}$$

(B) Define

$$\Delta_k := \sum_i a_i^k - \sum_j b_j^k. \tag{3}$$

(a) Let Δ_k be of "degree" k. Then resolution (1) implies

(∗) the Chern class $c_i(\mathcal{I}_X)$ is a degree-i polynomial in the Δ's.

In fact, $c_2(\mathcal{I}_X) = \frac{1}{2}\Delta_2, c_3(\mathcal{I}_X) = \frac{1}{3}\Delta_3,\ c_4(\mathcal{I}_X) = \frac{1}{4}\Delta_4 + \frac{1}{8}\Delta_2^2$, and $c_5(\mathcal{I}_X) = \frac{1}{5}\Delta_5 + \frac{1}{6}\Delta_2\Delta_3$.

Let $a_{r+1} + a_{r+2} = 29r, b_{-2} = b_{-1} = b_0 = 10r$, and

$$a_i = b_i + 1,\ i = 1, \ldots, r, \tag{4}$$

$$b_i = 10r + 2c_i, \quad \text{where} \quad r \leq c_1 \leq c_2 \leq \cdots \leq c_r \leq 2r. \tag{5}$$

(A result in [Ch1] implies that X is nonsingular.)

(b) Combining (3), (4) and (5), we have

$$\Delta_j = \sum_{h+k=j-1} s_h(\bar{c})p_k(r) + p_j(r) \tag{6}$$

where $s_h(\bar{c}) := \sum_i c_i^h$ and $p_k(r) \in \mathbb{Q}[r]$ is a polynomial of degree k, for $k = 1, \ldots, j$.

In particular, we have

$$\Delta_j = O(r^j) \tag{7}$$

and

$$c_i(\mathcal{I}_X) = O(r^i). \tag{8}$$

(c) Intersection theory gives

$$c_i(\mathcal{I}_X) = H^{5-i}(K + 6H)^{i-2}. \tag{9}$$

Combining (8) and (9), we have

$$H^3 = O(r^2), H^2K = O(r^3), HK^2 = O(r^4), K^3 = O(r^5). \tag{10}$$

(C) The intersection numbers H^iK^j are functions of r and $s_j(\bar{c})$. Indeed, this is a consequence of (6), (∗), and (9).

With (C) in mind, we set the following

Notation. $A \equiv B$, if $A - B$ is a function of r and $s_j(\bar{c})$. Here

$$s_j(\bar{c}) = \sum_i c_i^j.$$

(D) Substituting (4) and (5) into (2) and simplifying, we have

$$\dim H \equiv \sum_{i>j} \binom{2c_i - 2c_j + 4}{3}. \tag{11}$$

So we reduce our problem to the following:

For $r \gg 0$, varying $r \le c_1 \le \cdots \le c_r \le 2r$ while $s_j(\bar{c}) = \sum c_i^j$ being fixed, how many different values can $\sum_{i>j}(c_i - c_j)^3 + 2(c_i - c_j)$ or $2\sum ic_i^3 + 6\sum_{i>j} c_i c_j^2 + 2\sum ic_i$ take?

Now part (ii) of the Theorem follows from (10) and a small variation of the following proposition which will be proved in the next section.

Proposition. *Let $A = A_{(s_1,\ldots,s_5)} = \{(c_1,\ldots,c_r) | r \le c_1 \le \cdots \le c_r \le 2r, c_i \in \mathbb{N}, s_j(\bar{c}) := \sum c_i^j = s_j$ for $j = 1,\ldots,5\}$, and let $g = (g_1,\ldots,g_4) : A \to \mathbb{N}^4$ be defined by $g_j(\bar{c}) = \sum ic_i^j$ for $j = 1,2,3$ and $g_4(\bar{c}) = \sum_{i>j} c_i^2 c_j$. Then for $r \gg 0$, there is a box $B(r)$ of size $r^2 \times \cdots \times r^6$ in $\mathbb{N}^5$ such that for any $S = (s_1,\ldots,s_5)$ in $B(r)$, there is a box $B(S)$ in $\mathbb{N}^4$ of size $r^3 \times r^4 \times r^5 \times r^5$ such that the image of g contains $B(S)$.*

Remark 2. Exactly the same construction for surfaces in $\mathbb{P}^4$ gives $H^2 = O(r^2), HK = O(r^3), K^2 = O(r^4)$ for (10), and $\dim H \equiv \sum_{i>j} \binom{2c_i - 2c_j + 3}{2}$ for (11), hence part (i) of the Theorem.

Remark 3. In [Ch2] where we showed the number of components of the Hilbert schemes of some of such projectively normal 3-folds X is at least $y^{(y^{\frac{1}{5}})}$, where $y = K^3$, we have seen that the line bundle $\mathcal{O}_X(1)$ and its sections deform along with X. Also a quintic hypersurface section $Y = X \cap 5H$ deforms along with X. These give parts (iii) and (iv) of the Theorem.

Remark 4. It is plausible that one could show $n(K^2, c_2)$ and $n(K^3, c_1c_2, c_3)$ can be arbitrarily large (in terms of the Chern numbers) by considering Y to be the intersection of suitably many hypersurfaces with a projectively Cohen–Macaulay subvariety X of codimension 2 in $\mathbb{P}^n$ for odd $n \gg 0$, where the ideal defining X has a similar resolution to the defining ideal of the variety considered here. The same combinatorics show that the number of distinct dimensions of components of Hilbert scheme parametrizing subvarieties of $\mathbb{P}^n$ with the same Hilbert polynomial as X is in the order of r^n, while [Ch1] implies that the singularity of X has expected dimension (hence Y is nonsingular) and the adjunction formula implies that $K_Y^2 = O(r^4)$ and $c_2(Y) = O(r^4)$.

§2

In this section, we will prove the Proposition stated in §1. Our approach is to study the function $f = (s_1,\ldots,s_5,g_1,\ldots,g_4)$ rather than the sequence $(g_1,\ldots,g_4)$. After showing that the image of f contains a large enough box, we study the fibers over a point $(s_1,\ldots,s_5)$. For this study, it is more natural to first consider f as a function of continuous functions (rather than sequences, which are just discrete functions on a finite set), and then to approximate the integral points in the images of integral points.

Hence the following lemma is the key.

Lemma 1. *Let C^1 be the set of continously differentiable functions, and let $F = (F_1, \dots, F_9)\colon\ C^1 \to \mathbb{R}^9$ be an operator defined by*

$$F_j(\varphi) = \begin{cases} \int_r^{2r} \varphi^j(x)dx, j = 1, \dots, 5, \\ \int_r^{2r} x\varphi^{j-5}(x)dx, j = 6, 7, 8, \\ \int_{2r \geq x > y \geq r} \varphi^2(x)\varphi(y)dxdy,\ \ j = 9. \end{cases}$$

Then for any $K \gg 0$, there is a set S consisting of increasing functions, and a box Q_K in $\mathbb{R}^9$ of size $d_1 rK \times \cdots \times d_5 rK^5 \times d_6 r^2 K \times \cdots \times d_8 r^2 K^3 \times d_9 r^2 K^3$, where d_i's are constants independent of r and K such that $F(S) \supset Q_K$.

Proof. Define $f : C^1 \to \mathbb{R}^9$ by

$$f(\varphi) = \left(\int_1^2 \varphi dx, \dots, \int_1^2 \varphi^5 dx, \int_1^2 x\varphi dx, \dots, \right. \\ \left. \int_1^2 x\varphi^3 dx, \int_{2 \geq x > y \geq 1} \varphi^2(x)\varphi(y)dx, dy \right).$$

Let $D_\psi f_j(\varphi) = \lim\limits_{t \to 0} \frac{f_j(\varphi + t\psi) - f_j(\varphi)}{t}$ be the directional derivative of f_j in the direction of ψ at φ.

(α) If $D_\psi f_j(\varphi) = \langle \nabla f_j(\varphi), \psi \rangle = \int \nabla f_j(\varphi) \cdot \psi$, where ∇f_j is the gradient of f_j identified in the dual space, and

(β) if there is φ such that $\nabla f_1(\varphi), \dots, \nabla f_9(\varphi)$ are linearly independent (these are $j\varphi^{j-1}$ for $j = 1, \dots, 5, xj\varphi^{j-1}$ for $j = 6, 7, 8$ and $\nabla f_9(\varphi) = 2\varphi \int_x^2 \varphi(y)dy + \int_1^2 \varphi^2(y)dy$), then there are $\psi_1, \dots, \psi_9$ such that

$$D_{\psi_i} f_j(\varphi) = \delta_{ij}. \tag{12}$$

Now define $\bar{f} : \mathbb{R}^9 \to \mathbb{R}^9$

$$\bar{f}_j(t_1, \dots, t_9) = f_j(\varphi + t_1\psi_1 + \cdots + t_9\psi_9).$$

(Note that $\bar{f}$ factors through C^1 via the map $g\colon\ \mathbb{R}^9 \to C^1$ defined by

$$g(t_1, \dots, t_9) = \varphi + t_1\psi_1 + \cdots + t_9\psi_9,$$

i.e. we have $\bar{f}_j = f_j \circ g$.)

It is easy to see that

$$\frac{\partial \bar{f}_j}{\partial t_i}(0) = D_{\psi_i} f_j(\varphi) = \delta_{ij}.$$

So the implicit function theorem implies that the image of $\bar{f}$ contains a ball $\bar{B}$, say, of size $d_1 \times \cdots \times d_9$ in $\mathbb{R}^9$. Let B be a ball in $\mathbb{R}^9$ that contains the preimage of $\bar{B}$, i.e. $\bar{f}(B) \supset \bar{B}$.

To satisfy conditions (α) and (β), we take $\varphi = \sin \frac{x}{10}$. To make sure Q_K is mapped by increasing functions, we take ψ_i's such that $\frac{d\psi_i}{dx}$ is much smaller than $\frac{d\varphi}{dx}$. The set S is $g(B)$. Clearly enlarging the ball B by K (i.e. letting $\bar{\varphi} = K\varphi$) gives the image of size $d_1 K \times \cdots \times d_5 K^5 \times d_6 K \times \cdots \times d_8 K^3 \times d_9 K^3$, and rescaling (i.e. letting $\bar{x} = rx$) gives the factor r or r^2 in each of the coordinates. $\square$

Lemma 2. *Let* $B_r = \{(c_{\frac{r}{2}+1} \dots , c_r) | r \le c_{\frac{r}{2}+1} \le \dots \le c_r \le 2r, c_i \in \mathbb{N}\}$ *and* $f\colon B_r \to \mathbb{N}^9$ *defined by* $f = (s_1, \dots , s_5, t_1, \dots , t_3, u)$*, where* $s_j(\bar{c}) = \sum c_i^j, t_j(\bar{c}) = \sum i c_i^j$*, and* $u(\bar{c}) = \sum\limits_{i>j} c_i^2 c_j$*. Then there is a box* Q_r *of size* $r^2 \times \dots \times r^6 \times r^3 \times r^4 \times r^5 \times r^5$ *in* $\mathbb{N}^9$ *such that every integral point in* Q_r *lies together with the image under* f *of an integral point of* B_r*, in a box of size* $r \times \dots \times r^5 \times r^2 \times r^3 \times r^4 \times r^4$*.*

Proof. In Lemma 1, take $c_{\frac{r}{2}+k} = [\varphi(r+2k)]$, the integral part of $\varphi(r+2k)$. □

Lemma 3. *Let* $A_r = \{(c_1, \dots , c_r) | r \le c_1 \le \dots \le c_r \le 2r, c_i \in \mathbb{N}\}$ *and* f *be as in Lemma 2. Then for* $r \gg 0$*, there is a box* Q_r *of size* $r^2 \times \dots \times r^6 \times r^3 \times r^4 \times r^5 \times r^5$ *such that every integral point in* Q_r *is approximated by the image of an integral point up to arbitrarily small power of* r*.*

Proof. Take $r_1 = r, r_2, \dots$ such that $r_i > r_{i+1} > r_i^{\frac{5}{6}}$ and apply Lemma 2 for each r_i, then take the Cartesian product of all B_{r_i}'s. (Note: if $(\bar{c}, \bar{d}) \in B_{r_i} \times B_{r_k}$ for $i > k$, then $s_j(\bar{c}, \bar{d}) = s_j(\bar{c}) + s_j(\bar{d}) \quad u(\bar{c}, \bar{d}) = u(\bar{c}) + u(\bar{d}) + s_2(\bar{c}) s_1(\bar{d}), t_j(\bar{c}, \bar{d}) > t_j(\bar{c}) + t_j(\bar{d})$.) □

To conclude the proof of the Proposition, we notice that $A_(s_1, \dots , s_5)$ is the fiber of A_r in Lemma 3 over a point $(s_1, \dots , s_5)$.

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School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540

E-mail address: mcc@math.ias.edu

Current address: Department of Mathematics, University of California, Riverside, California 92521

E-mail address: mcc@math.ucr.edu