PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 9, September 1997, Pages 2779–2787
S 0002-9939(97)03877-X

GENERALIZED PRINCIPAL SERIES REPRESENTATIONS OF $SL(1+n,\mathbb{C})$

ANTHONY H. DOOLEY AND GENKAI ZHANG

(Communicated by Palle E. T. Jorgensen)

Abstract. We consider certain induced representations of the group $SL(n+1,\mathbb{C})$ realized on line bundles over the projective space of $\mathbb{C}^{n+1}$. We calculate the intertwining operators in the compact picture. We find all the unitarizable representations and determine the invariant norm.

§0. Introduction

Let $SL(2,\mathbb{C})$ be the group of complex 2×2 matrices with determinant one. Its irreducible unitary representations have been classified [Kn]. They are either principal series representations or Stein's complementary series, and they are all induced representations from a parabolic subgroup. In the present paper we will consider similar representations of $SL(1+n,\mathbb{C})$.

The above representations of $SL(2,\mathbb{C})$ can be realized as follows. Consider the unit disk in the complex plane. Its group of biholomorphic mappings is $G = SL(2,\mathbb{R})$. Let $\mathfrak{g}$ be its Lie algebra and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be its Cartan decomposition. The Lie algebra $\mathfrak{g}^{\mathbb{C}}$ of $G^{\mathbb{C}}=SL(2,\mathbb{C})$ has a Harish-Chandra decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^{+}+\mathfrak{k}^{\mathbb{C}}+\mathfrak{p}^{-}$. Let P be the Lie subgroup of $SL(2,\mathbb{C})$ with Lie algebra $\mathfrak{k}^{\mathbb{C}}+\mathfrak{p}^{-}$. Take a linear functional ν on $i\mathfrak{k}$ and l on $\mathfrak{k}$ and consider the induced representation $Ind_P^{G^{\mathbb{C}}}(l\otimes\nu\otimes 1)$ from P. The unitary principal series representations are those with ν purely imaginary and $l=0$; the Stein complementary series are, after a normalization, $0<\nu<2$ and $l=0$. Moreover they exhaust all the unitary representations of $SL(2,\mathbb{C})$.

We can replace the unit disk by any irreducible bounded symmetric domain G/K and consider the corresponding induced representation. Now $\mathfrak{k}$, the Lie algebra of K, has one-dimensional center $\mathfrak{z}$. We take a linear functional ν on $i\mathfrak{z}$ and a linear functional l on $\mathfrak{z}$ and consider the induced representation $Ind_P^{G^{\mathbb{C}}}(l\otimes\nu\otimes 1)$ of $G^{\mathbb{C}}$. When the bounded domain is the hyperbolic ball of $n\times n$ complex matrices with operator norm less than 1, the group $G^{\mathbb{C}}$ is then $SL(2n,\mathbb{C})$ and the above representation is studied by Stein [St]. It is proved that there are complementary series when $l=0$. Moreover later it turns out that Stein's complementary series play an important role in the classification of all the unitary representations of $SL(2n,\mathbb{C})$; see [Vo1] and [Vo2].

Received by the editors September 26, 1995 and, in revsied form, April 9, 1996.
1991 *Mathematics Subject Classification.* Primary 22E46, 43A90.
Key words and phrases. Representation, semisimple Lie group.
Research sponsored by the Australian Research Council.

©1997 American Mathematical Society

In this paper we will consider the case when the bounded symmetric domain is the unit ball in $\mathbb{C}^n$. Let G^* be the compact dual of G. The induced representation $\pi_{l,\nu} = Ind_P^{G^{\mathbb{C}}}(l \otimes \nu \otimes 1)$ can be realized on $L^2(G^*/K, l)$, the space of L^2-sections of a line bundle on G^*/K induced from the one-dimensional representation l of K. Under G^* it decomposes into $V^{\underline{\mathbf{m}}}$ with multiplicity one, and each $V^{\underline{\mathbf{m}}}$ contains vector $\phi_{l,\underline{\mathbf{m}}}$ of type l, unique up to scalar multiplication.

Let Z' be an element in the center of $i\mathfrak{k}$. Then $\pi_{l,\nu}(Z')$, the infinitesimal action of Z', maps $\phi_{l,\underline{\mathbf{m}}}$ to a linear combination of $\phi_{l,\underline{\mathbf{m}}'}$. We will find all the coefficients of $\phi_{l,\underline{\mathbf{m}}'}$ appearing in this linear combination. Many results then follow from the expansion.

The generalized principal series representations for other groups have also been studied in [HT], [KS], [Jo], [OZ] and [Zh2].

In §1 we introduce the Lie algebra of Hermitian type and the Harish-Chandra decomposition of its complexification and the corresponding parabolic subalgebra. The parabolically induced representation can be realized on the L^2-space of sections of a certain line bundle over the compact Hermitian symmetric space. The decomposition into irreducibles of the L^2-space is done in §2 by using Schlichtkrull's generalization of the Helgason theorem. In §3 we study the infinitesimal action of the Lie algebra of the Harish-Chandra module. We restrict ourselves to the rank one Hermitian symmetric space and find all the coefficients in the expansion of the infinitesimal action of Z' on the spherical polynomials of l-type. In §4 we examine the existence of the complementary series.

We plan to study the similar generalized principal series representation for some higher rank Hermitian symmetric spaces in a subsequent paper.

§1. Preliminaries and notation

The material of this section is well known; we refer the reader to [KW] and [Sch] for details.

Let $\mathfrak{g}$ be a simple Lie algebra of Hermitian type and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Thus $\mathfrak{k}$ has one-dimensional center $\mathfrak{z}$. Choose $Z_0 \in \mathfrak{z}$ so that $\mathrm{ad}(Z_0)$ has eigenvalue $0, \pm i$. Let $\mathfrak{p}^{\pm}$ be the eigenspaces of $\mathrm{ad}(Z_0)$ in $\mathfrak{p}^{\mathbb{C}}$ with eigenvalues ± 1. We thus have the Harish-Chandra decomposition of $\mathfrak{g}^{\mathbb{C}}$, the complexification of $\mathfrak{g}$, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{p}^- + \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^+$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$ let and $\gamma_1 < \cdots < \gamma_r$ be the Harish-Chandra strongly orthogonal roots. Let $e_{\pm} \in \mathfrak{p}^{\pm}$ be the corresponding root vectors e_j such that $[e_j, e_{-j}] = -D_j$ and $[D_j, e_j] = 2e_j$. Thus $D_j \in \mathfrak{h}$ and $\gamma_j(D_k) = 2\delta_{j,k}$, where $\delta_{j,k}$ is the Kronecker symbol. We put $e = e_1 + \cdots + e_r$, $e_- = e_{-1} + \cdots + e_{-r}$ and $\xi_j = e_j - e_{-j}$. Thus $\xi_j \in \mathfrak{p}$ and they span a maximal abelian subspace $\mathfrak{h}_{\mathfrak{p}}$ of $\mathfrak{p}$. We let $\beta_j \in \mathfrak{h}_{\mathfrak{p}}^*$ be defined by $\beta_j(\xi_k) = 2\delta_{j,k}$.

Denote by $n = \dim \mathfrak{p}^+$, $p = \mathrm{tr}_{\mathfrak{p}^+} \mathrm{ad} D_1$. Then p is an integer and is called the genus of $(\mathfrak{g}, \mathfrak{k})$. ($\mathrm{tr}_{\mathfrak{p}^+}$ is the trace functional on the complex space $\mathfrak{p}^+$.) We define $Z = \frac{p}{n} Z_0 \in \mathfrak{z}$. Thus $\mathfrak{z} = \mathbb{R}Z$. We let further $Z' = -iZ$, and $\mathfrak{a} = \mathbb{R}Z'$. The centralizer of $\mathfrak{a}$ in $\mathfrak{g}^{\mathbb{C}}$ is then $\mathfrak{k}^{\mathbb{C}}$.

Let $G^{\mathbb{C}}$ be a connected group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$, and G^*, $K^{\mathbb{C}}$, K, A and P^- the subgroups with Lie algebras $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p}$, $\mathfrak{k}^{\mathbb{C}}$, $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{p}^-$ respectively. G^* is a maximal compact subgroup $G^{\mathbb{C}}$ and $K^{\mathbb{C}}P^-$ is the parabolic subgroup corresponding to $\mathfrak{a}$.

Each $l \in \mathbb{Z}$ defines a linear functional on $\mathfrak{z}$, $Z \mapsto l$, and this extends further to a representation τ_l of K. Our main interest is the induced representation $\pi_{l,\nu} = Ind_P^{G^{\mathbb{C}}}(l \otimes \nu)$ where ν is a linear functional on $\mathfrak{a}$. It can be realized on the space of functions f on $G^{\mathbb{C}}$ such that, writing elements in $K^{\mathbb{C}}$ as $\exp(tZ)\exp(sZ')k$ where k is in the semisimple part of $K^{\mathbb{C}}$,

(1) $f(g\exp(tZ)\exp(sZ')kp) = e^{-tl}e^{-s\nu(Z')}f(g)$ $(p \in P^-)$; and
(2) $f\big|_{G^*} \in L^2(G^*)$.

and the group $G^{\mathbb{C}}$ by the left regular action.

The representation $\pi_{l,\nu}$ can be realized on the space $L^2(G^*/K, l)$ of all functions $f \in L^2(G^*)$ such that $f(g\exp(tZ)k) = e^{-tl}f(g)$.

Remark. When $(\mathfrak{g}, \mathfrak{k})$ is a Hermitian pair of tube type the central element $Z' = \frac{1}{r}\sum_{j=1}^r D_j$; see [Sch]. However for non-tube type domains, $\sum_{j=1}^r D_j$ is not in the center. This fact means that many representation theoretic properties differ from tube-domain to non-tube domain; see e.g. Theorem 4.1 below.

§2. The L^2-space of sections of line bundles over G^*/K.

The space $L^2(G^*/K, l)$ is decomposed under G^* into $V \otimes V^*_{K,l}$, where V are irreducible modules so that $V^*_{K,l} \neq 0$. Here V^* is the contragradient of V and $V^*_{K,l}$ is its subspace of type $-l$ under K; see[He, Ch. V, Theorem 3.5]. It follows from the Schlichtkrull's generalization of Helgason theorem [Sch, Theorem 7.2] that those irreducibles are of the form $V^{\underline{\mathbf{m}}}$ with highest weight $\underline{\mathbf{m}} = \frac{1}{2}(m_1\beta_1 + \cdots m_r\beta_r)$, with conditions $|l| \leq m_1 \leq \cdots \leq m_r$ and $|l| \cong m_1 \cong \cdots \cong m_r \bmod 2$. Moreover each $V^{\underline{\mathbf{m}}}$ contains a unique one dimensional space of K-type l. Thus $L^2(G^*/K, l)$ is decomposed into $V^{\underline{\mathbf{m}}}$ with multiplicity one.

Now again by [Sch, Theorem 7.2], each irreducible $V^{\boldsymbol{\mu}}$ contains a unique l-spherical vector $\phi_{l,\boldsymbol{\mu}}$ such that $\phi_{l,\boldsymbol{\mu}}(k_1gk_2) = \tau_l(k_1)^{-1}\phi_{l,\boldsymbol{\mu}}(g)\tau_l(k_2)^{-1}$ and that $\phi_{l,\boldsymbol{\mu}}(e) = 1$.

When $l = 0$ and $\underline{\mathbf{m}} = \beta_r$ we write $\phi_{l,\boldsymbol{\mu}} = \phi_{\beta_r}$. In particular,

$$\phi_{\beta_r}(\exp(H)) = \frac{1}{r}\sum_{j=1}^r \operatorname{ch}\beta_j(H), \quad H \in \mathfrak{h}_{\mathfrak{p}}. \tag{2.1}$$

Example. We take $\mathfrak{g}^{\mathbb{C}}$ to be $\mathfrak{sl}(2,\mathbb{C})$ and write it as $\mathbb{C}e_- + \mathbb{C}D + \mathbb{C}e_+$ with $[D, e_\pm] = \pm 2e_\pm$ and let $G^* = SU(2)$ be a real form of $SL(2,\mathbb{C})$ with $K = \exp(iD)$ its Cartan subgroup. Each irreducible representation of $SU(2)$ is of highest weight $\frac{m}{2}D^*$. It can be realized on the space of homogeneous polynomials on $\mathbb{C}^2$ of degree m. Thus fixing a basis of $\mathbb{C}^2$ we have $z_1^{m-j}z_2^j$ are all the weight vectors, of weight $\frac{m-2j}{2}D^*$, $j = 1, \cdots, m$. For any l, the weight $\frac{l}{2}D^*$ appears in the representation if and only if $|l| \leq m$ and $l \cong m$ (mod 2) or equivalently $m = l + 2j$ for some $j = 0, 1, \cdots, m$. The corresponding weight vector is $z_1^{m-j}z_2^j$. If we take the matrix coefficient with respect to this vector we get the spherical polynomials of one-dimensional K-type, which are certain Jacobi polynomials [PZ].

§3. Induced representations of $G^{\mathbb{C}}$

Let $X(l,\nu)$ be the space of K-finite functions in $L^2(G^*/K, l)$ and $X(l,\nu)^K$ be the subspace consisting of functions which transform under K from both sides according to the character τ_l. It follows from §2 that $X(l,\nu)^K = \sum_{\underline{\mathbf{m}}} \mathbb{C}\phi_{l,\underline{\mathbf{m}}}$ Now

$\pi_{l,\nu}(Z')$, the infinitesimal action of Z' on $X(l,\nu)$, maps $X(l,\nu)^K$ into itself, since Z' is in the center of $K^{\mathbb{C}}$ and commutes with K. In this section we will calculate this action.

Lemma 3.1. *Let $h = \exp(H) \in \exp(\mathfrak{h}_{\mathfrak{p}}^{\mathbb{C}})$, $H \in \mathfrak{h}_{\mathfrak{p}}^{\mathbb{C}}$. Then modulo $(\mathfrak{k} \ominus \mathfrak{z})^{\mathbb{C}} + \mathfrak{p}^-$ in $\mathfrak{g}^{\mathbb{C}}$*

$$Ad(h^{-1})Z' = \left(1 - \frac{rp}{2n} + \frac{p}{2n}\sum_{j=1}^{r} \operatorname{ch}\beta_j(H)\right)Z' + \frac{p}{2n}(\sum_{j=1}^{r} \operatorname{sh}\beta_j(H)\xi_j).$$

Proof. Write $H = \sum_{j=1}^r c_j\xi_j$ with $c_j \in \mathbb{C}$. Thus

$$\mathrm{Ad}(h^{-1})Z' = \mathrm{Ad}(\exp(-c_r\xi_r)) \cdots \mathrm{Ad}(\exp(-c_1\xi_1))Z'.$$

Now $\mathrm{Ad}(\exp(-c_j\xi_j))Z' = \exp(\mathrm{ad}(-c_j\xi_j)(Z'))$ and recalling (1.1), we get

$$\mathrm{ad}(-c_j\xi_j)(Z') = [-c_j\xi_j, Z'] = \frac{p}{2n}(2c_j)(e_j + e_{-j}),$$

$$(\mathrm{ad}(-c_j\xi_j))^2(Z') = \frac{p}{2n}(2c_j)^2 D_j,$$

$$(\mathrm{ad}(-c_j\xi_j))^3(Z') = \frac{p}{2n}(2c_j)^3(e_j + e_{-j}),$$

and so on.

It follows that

$$\mathrm{Ad}(\exp(-c_j\xi_j))Z' = Z' + \frac{p}{2n}\operatorname{sh}(2c_j)(e_j + e_{-j}) + \frac{p}{2n}(\operatorname{ch}(2c_j) - 1)D_j.$$

Moreover $\{e_j, e_{-j}, D_j\}$ are n-commuting $\mathfrak{sl}(2,\mathbb{C})$-triple; see [KS]. Thus, we have

$$\begin{aligned} Ad(h^{-1})\xi_e &= Z' + \frac{p}{2n}\sum_{j=1}^{r}\operatorname{sh}(2c_j)(e_j + e_{-j}) + \frac{p}{2n}\sum_{j=1}^{r}(\operatorname{ch}(2c_j) - 1)D_j \\ &= Z' + \frac{p}{2n}\sum_{j=1}^{r}\operatorname{sh}(2c_j)(e_j - e_{-j}) + \frac{p}{n}\sum_{j=1}^{r}\operatorname{sh}(2c_j)e_{-j} \\ &\qquad + \frac{p}{2n}\sum_{j=1}^{r}\operatorname{ch}(2c_j)D_j - \frac{p}{2n}\sum_{j=1}^{r}D_j. \end{aligned}$$

We write $D_j = Z' + D'_j$ and $\mathrm{tr}_{\mathfrak{p}^+}\,\mathrm{ad}D'_j = \mathrm{tr}_{\mathfrak{p}^+}\,\mathrm{ad}D_j - \mathrm{tr}_{\mathfrak{p}^+}\,\mathrm{ad}Z' = p - p = 0$. Thus $D'_j \in (\mathfrak{k} - \mathfrak{z})^{\mathbb{C}}$. This concludes the proof. □

We now calculate the infinitesimal action of Z' on $\phi_{l,\boldsymbol{\mu}}$. By definition

$$\begin{aligned} \pi_{l,\nu}(Z')\phi_{l,\boldsymbol{\mu}}(h) &= \frac{d}{dt}\phi_{l,\boldsymbol{\mu}}(\exp(-tZ')h)\Big|_{t=0} = \frac{d}{dt}\phi_{l,\boldsymbol{\mu}}(h\exp(-tAd(h^{-1})Z'))\Big|_{t=0} \\ &= (1 - \frac{rp}{2n} + \frac{p}{2n}\sum_{j=1}^{r}\operatorname{ch}\beta_j(H))\nu(Z')\phi_{l,\boldsymbol{\mu}}(h) + \frac{p}{2n}\sum_{j=1}^{n}\operatorname{sh}\beta_j(H)\partial_{\xi_j}\phi_{l,\boldsymbol{\mu}}(h) \\ &= (1 - \frac{rp}{2n})\nu(Z')\phi_{l,\boldsymbol{\mu}}(h) + \frac{p}{2n}\nu(Z')(\sum_{j=1}^{r}\operatorname{ch}\beta_j(H))\phi_{l,\boldsymbol{\mu}}(h) \\ &\quad + \frac{p}{2n}\sum_{j=1}^{n}\operatorname{sh}\beta_j(H)\partial_{\xi_j}\phi_{l,\boldsymbol{\mu}}(h), \end{aligned}$$

and the second term above can be written as $\frac{rp}{2n}\nu(Z')\phi_{\beta_r}\phi_{l,\boldsymbol{\mu}}$ by (2.1).

From now on we assume that $(\mathfrak{g},\mathfrak{k})$ is the rank one symmetric pair $(\mathfrak{su}(1,n),\mathfrak{u}(n))$. The space G^*/K is then the projective space of $\mathbb{C}^{n+1}$. Thus the highest weights appearing in $L^2(G^*/K,l)$ are of the form $\boldsymbol{\mu}=\frac{m}{2}\beta$ with $|l|\le m$ and $l\cong m \bmod 2$.

Proposition 3.2. *The following recurrence formulas hold:*

$$\begin{aligned}\phi_\beta(g)\phi_{l,\frac{m}{2}\beta}(g) = \frac{1}{2}b^+_{l,m}\phi_{l,\frac{m+2}{2}\beta}(g) + \frac{1}{2}b^-_{l,m}\phi_{l,\frac{m-2}{2}\beta}(g)\\ + \frac{(l+1-n)(l-1+n)}{(m+n+1)(m+n-1)}\phi_{l,\frac{m}{2}\beta}(g),\ g\in G^*,\end{aligned}$$

where

$$b^+_{l,m} = \frac{(m+2n+l)(m+2n-l)}{(m+n)(m+n+1)},$$

$$b^-_{l,m} = \frac{(-m+l)(-m-l)}{(m+n)(m+n-1)}.$$

Moreover $b^+_{l,m}$ (resp. $b^-_{l,m}$) is nonzero if and only if $\frac{m+2}{2}\beta$ (resp. $\frac{m-2}{2}\beta$) is not a highest weight.

Proof. When $l=0$ then $\phi_{l,\frac{m}{2}\beta}=\phi_{\frac{m}{2}\beta}$. This was proved by Vretare; see [V1], [V2] and [Zh1]. For general l it is essentially proved in [Hec, Corollary 7.7] using algebraic methods. We give here a proof using again the idea of Vretare.

We first show that the product $\phi_\beta(g)\phi_{l,\frac{m}{2}\beta}(g)$ can indeed be expressed in terms of $\phi_{l,\frac{m}{2}\beta\pm\beta}(g)$. As in [Zh1] we consider the tensor product $V^{\frac{m}{2}\beta}\otimes\mathfrak{g}^{\mathbb{C}}$, where $V^{\frac{m}{2}\beta}$ is a module of G^* with highest weight $\frac{m}{2}\beta$ and $\mathfrak{g}^{\mathbb{C}}$ is of highest weight β. The spherical polynomials $\phi_{l,\frac{m}{2}\beta}$ are then the matrix coefficients with respect to the corresponding vector, say $v_{l,\frac{m}{2}\beta}$ of one dimensional K-type l; ϕ_β is the matrix coefficient with respect to the K-invariant vector v_β $(=Z')$ in $\mathfrak{g}^{\mathbb{C}}$. Thus the matrix coefficient of $v_{l,\frac{m}{2}\beta}\otimes v_\beta$ is $\phi_{l,\frac{m}{2}\beta}\phi_\beta$.

Now each component of $v_{l,\frac{m}{2}\beta}\otimes v_\beta$ in the tensor product decomposition of $V^{\frac{m}{2}\beta}\otimes\mathfrak{p}$ must be of K-type l. However those containing a vector of K-type l must have highest weights $\frac{m}{2}\beta\pm\beta$ or $\frac{m}{2}\beta$, by the weight space decomposition of $\mathfrak{g}^{\mathbb{C}}$ and [Sch, Theorem 7.2]. (Note that for the case considered in [OZ] where the root system is of type A, tensor product arguments show that the term $\phi_{l,\frac{m}{2}\beta}$ does not appear; whereas in our case the root system is of type C and indeed $\phi_{l,\frac{m}{2}\beta}$ will appear in the expansion for most m, as we will see.) By taking the matrix coefficients of the decomposition of the vector $\phi_{l,\frac{m}{2}\beta}\otimes\phi_\beta$ we get an expansion of $\phi_\beta(g)\phi_{l,\frac{m}{2}\beta}(g)$ in terms of $\phi_{l,\frac{m}{2}\beta\pm\beta}(g)$ and $\phi_{l,\frac{m}{2}\beta}$.

Next we find the coefficient in the expansion. We recall [Shi] that we can actually define the l-spherical function $\Phi_{l,\lambda}$ (denoted by $\phi_{l,\lambda}$ in [Sch]) on the non-compact dual G and when $\lambda=\frac{m}{2}\beta+\rho$, $\Phi_{l,\frac{m}{2}\beta+\rho}=\phi_{l,\frac{m}{2}\beta}$ and we work on the expansion, with $\lambda\in\mathfrak{a}^*_{\mathbb{C}}$

$$\phi_\beta(g)\Phi_{l,\lambda}(g) = c_{l,\beta}(\lambda)\Phi_{l,\lambda+\beta}(h) + c_{l,-\beta}(\lambda)\Phi_{l,\lambda-\beta}(h) + \text{Remainder}$$

for some numbers $c_{l,\pm\beta}(\lambda)$. Here the term Remainder, if it appears, is a finite sum of spherical functions $\Phi_{l,\lambda'}$ with λ' satisfying $\lambda'<\lambda\pm\beta$ in the ordering of $\mathfrak{a}^*_+$ (i.e., $\beta>0$). Take $H_0\in\mathfrak{a}$ such that $\beta(H_0)>0$ and let $h=\exp(tH_0)$. Multiply both sides in the expansion by $e^{t(\rho-\lambda)(H_0)}$ and take the limit as $t\to\infty$. Now using the limit formula in [Shi, Proposition 4.9] (see also [Zh3, Corollary 2.2]), we have with $\lambda = \frac{m}{2}\beta + \rho$,

$$c_{l,\pm\beta}(\lambda) = C(\beta+\rho)\frac{C_l(\pm(\frac{m}{2}\beta+\rho))}{C_l(\pm(\frac{m}{2}\beta+\rho\pm\beta))} = \frac{1}{2}\frac{C_l(\pm(\frac{m}{2}\beta+\rho))}{C_l(\pm(\frac{m}{2}\beta+\rho\pm\beta))},$$

where $C_l(\cdot)$ is the generalized Harish-Chandra C-function; see [Sch, Remark 7.3] and [Zh3, Corollary 2.2]. Using the formula given there we find the coefficient of $\phi_{l,\frac{m\pm2}{2}\beta}$. Now evaluating the recurrence formula at the identity and after a routine calculation we find the coefficient of $\phi_{l,\frac{m}{2}\beta}$. □

Proposition 3.3. *Let $h = \exp(H) \in \exp(\mathfrak{h}_{\mathfrak{p}}^{\mathbb{C}})$, $H \in \mathfrak{h}_{\mathfrak{p}}^{\mathbb{C}}$. Then*

$$\begin{aligned}\operatorname{sh}(\beta(H))\partial_\xi\phi_{l,\frac{m}{2}\beta}(h) = \frac{1}{2}m\, b_m^+\,\phi_{l,\frac{m+2}{2}}(h) + \frac{1}{2}(-m-2n)\, b_m^-\,\phi_{l,\frac{m-2}{2}\beta}(h)\\ - \frac{l^2(n+1)+m(n-1)(m-2n)}{(-1+m+n)(1+m+n)}\phi_{l,\frac{m}{2}\beta},\end{aligned}$$

where $b_m^\pm(k)$ are as in Proposition 3.2.

Proof. The idea of the proof is similar to that of Lemma 3.1. We sketch it here. By considering the tensor product as above we know that $\operatorname{sh}(\beta(H))\partial_\xi\phi_{l,\frac{m}{2}\beta}(h)$ can be expanded in terms of $\phi_{l,\frac{m\pm2}{2}\beta}(h)$ and $\phi_{l,\frac{m}{2}\beta}(h)$. Taking the limit of this expansion along $\mathfrak{h}_{\mathfrak{p}}^+$ and using the Weyl invariance as in Proposition 3.2 we find that that the coefficients of $\phi_{l,\frac{m+2}{2}}(h)$ and $\phi_{l,\frac{m-2}{2}\beta}(h)$ are $\frac{1}{2}mb_{l,m}^+$ and $\frac{1}{2}(-m-2n)b_{l,m}^-$ respectively. Now evaluating both sides at $H=0$ we find the coefficient of $\phi_{l,\frac{m}{2}\beta}$ (after some elementary algebra). □

Recall the formula for $\pi_{l,\nu}(Z')\phi_{l,\frac{m}{2}\beta}$ after Lemma 3.1. Using Propositions 3.2 and 3.3 we now obtain the following formula for $\pi_{l,\nu}(Z')\phi_{l,\frac{m}{2}\beta}$.

Theorem 3.4. *We have the following formula for the infinitesimal action*:

$$\begin{aligned}\pi_{l,\nu}(Z')\phi_{l,\frac{m}{2}\beta} &= \frac{1}{2}(\frac{n+1}{2n}\nu(Z')+m)b_{l,m}^+\phi_{l,\frac{m+2}{2}\beta}\\ &+\frac{1}{2}(\frac{n+1}{2n}\nu(Z')-m-2n)b_{l,m}^-\phi_{l,\frac{m-2}{2}\beta}\\ &+\frac{(l^2(n+1)+m(n-1)(m-2n))}{(-1+m+n)(1+m+n)}(\nu-\delta)(Z')\phi_{l,\frac{m}{2}\beta},\end{aligned}$$

where $b_{l,m}^\pm$ is given in Proposition 3.2.

§4. Intertwining operators and the complementary series

In this section we study the intertwining operator between pairs of representations of the complementary series, corresponding to those real ν for which the representation $\pi_{l,\nu}$ is unitarizable and irreducible.

Suppose A is an intertwining operator from $\pi_{l,\nu}$ to $\pi_{l',\nu'}$. Clearly $|l| = |l'|$. Furthermore A acts on each space with K-type $\frac{m}{2}\beta$ as a constant $A(\frac{m}{2}\beta)$, by Schur's lemma. Now

$$A\pi_{l,\nu}(Z')\phi_{l,\frac{m}{2}\beta} = \pi_{l',\nu'}(Z')A\phi_{l,\frac{m}{2}\beta} = A(\tfrac{m}{2}\beta)\pi_{l',\nu'}(Z')\phi_{l,\frac{m}{2}\beta}. \tag{4.1}$$

Using Theorem 3.4 and comparing the coefficients of $\phi_{l,\frac{m}{2}\beta}$ we find

$$\frac{(l^2(n+1)+m(n-1)(m-2n))}{(-1+m+n)(1+m+n)}(\nu-\delta)(Z')A(\frac{m}{2}\beta)$$
$$=\frac{(l^2(n+1)+m(n-1)(m-2n))}{(-1+m+n)(1+m+n)}(\nu'-\delta)(Z')A(\frac{m}{2}\beta).$$

If $n=1$ and $l=0$ both sides are 0. Otherwise if $n\neq 1$ or $l\neq 0$, this equation holds if and only if $\nu(Z')=\nu'(Z')$, i.e., $\nu=\nu'$. Suppose now that $\nu=\nu'$. Comparing the coefficients of $\phi_{l,\frac{m\pm 2}{2}\beta}$ in (4.1) gives

$$\frac{A(\frac{m-2}{2}\beta)}{A(\frac{m}{2}\beta)}=\frac{\tilde{\nu}'+m_j-\rho_j-\rho_n}{\tilde{\nu}+m_j-\rho_j-\rho_n}=1.$$

Thus A is the identity operator, which further implies that $l=l'$. Thus we have proved the following (somewhat surprising) result.

Theorem 4.1. *If $n\neq 1$ or $l\neq 0$, then there exists no intertwining operator from $X(l,\nu)$ to $X(l',\nu')$ unless $l=l'$ and $\nu=\nu'$, in which case the intertwining operator is the trivial one.*

Note that in most cases there exist intertwining operators between dual representations; see [Jo], [KS], [St], [OZ] and [Zh2].

We now consider complementary series, namely those real ν with $\nu(Z')\in\mathbb{R}$ for which $\pi_{l,\nu}$ are irreducible and unitarizable. Suppose $\pi_{l,\nu}$ is unitary with a unitary inner product $(\cdot,\cdot)$. Then $\pi_{l,\nu}(Z')$ is skew symmetric under the K-decomposition. In particular,

$$\langle\pi_{l,\nu}(Z')\phi_{l,\frac{m}{2}\beta},\phi_{l,\frac{m}{2}\beta}\rangle=-\langle\phi_{l,\frac{m}{2}\beta},\pi_{l,\nu}(Z')\phi_{l,\frac{m}{2}\beta}\rangle.$$

Theorem 3.4 and Schur's lemma give us, for all $m\in\mathbb{Z}$,

$$(\nu-\delta)(Z')\frac{l^2(n+1)+m(n-1)(m-2n)}{(-1+m+n)(1+m+n)}$$
$$=-\overline{(\nu-\delta)(Z')}\frac{l^2(n+1)+m(n-1)(m-2n)}{(-1+m+n)(1+m+n)}$$

and since $(\nu-\delta)(Z')$ is real we have

$$(\nu-\delta)(Z')\frac{l^2(n+1)+m(n-1)(m-2n)}{(-1+m+n)(1+m+n)}=0.$$

Thus either $\nu=\delta$ in which case $\pi_{l,\nu}$ is the irreducible and unitary principal series representation ([Kn]), or $l^2(n+1)+m(n-1)(m-2n)=0$ for all m. The left-hand side is a quadratic form in $m\in\mathbb{Z}$; thus all the coefficients of m^2, m, and the constant term vanish. This implies that $n=1$ and $l=0$.

Let $n=1$ and $l=0$. The Lie algebra in question is $\mathfrak{sl}(2,\mathbb{C})$. Suppose $A(\nu,\nu')$ is an intertwining operator from $\pi_{0,\nu}$ to $\pi_{0,\nu'}$. Then by Schur's lemma $A(\nu,\nu')$ acts on each space with K-type $\frac{m}{2}\beta$ as a constant $A(\nu,\nu')(\frac{m}{2}\beta)$. Thus

$$A(\nu,\nu')\pi_{0,\nu}(Z')\phi_{l,\frac{m}{2}\beta}=\pi_{0,\nu'}(Z')A(\nu,\nu')\phi_{l,\frac{m}{2}\beta}=A(\nu,\nu')(\frac{m}{2}\beta)\pi_{0,\nu'}(Z')\phi_{l,\frac{m}{2}\beta}.$$

Using Theorem 3.4 and comparing the coefficients of $\phi_{l,\frac{m\pm 2}{2}\beta}$ we find

$$A(\nu,\nu')(\frac{m+2}{2}\beta)(\frac{n+1}{2n}\nu(Z')+m)\frac{b^+_{l,m}}{2}=A(\nu,\nu')(\frac{m}{2}\beta)(\frac{n+1}{2n}\nu'(Z')+m)\frac{b^+_{l,m}}{2}$$

and

$$\begin{aligned}A(\nu,\nu')(\frac{m-2}{2}\beta)(\frac{n+1}{2n}\nu(Z')-m-2n)\frac{b^-_{l,m}}{2}\\ =A(\nu,\nu')(\frac{m}{2}\beta)(\frac{n+1}{2n}\nu'(Z')-m-2n)\frac{b^-_{lm}}{2}.\end{aligned}$$

Replacing $\frac{m}{2}\beta$ by $\frac{m-2}{2}\beta$ in the first equality and comparing with the second one we get

$$\frac{A(\nu,\nu')(\frac{m-2}{2}\beta)}{A(\nu,\nu')(\frac{m}{2}\beta)}=\frac{\frac{n+1}{2n}\nu'(Z')-m-2n}{\frac{n+1}{2n}\nu'(Z')-m-2n}=\frac{\frac{n+1}{2n}\nu(Z')+m-2}{\frac{n+1}{2n}\nu'(Z')+m-2}. \tag{4.2}$$

Simplifying the equality (recall that $n=1$), we obtain

$$(\tilde{\nu'}(Z')-\tilde{\nu}(Z')(\tilde{\nu'}(Z')+\tilde{\nu}(Z')-2)=0,$$

which gives $\tilde{\nu'}(Z')=\tilde{\nu}(Z')$ or $\tilde{\nu'}(Z')+\tilde{\nu}(Z')=2$. That is, there is an intertwining operator between $\pi_{0,\nu}$ and $\pi_{0,\nu'}$ only if they are dual.

Suppose now $\nu'(Z')=2-\nu(Z')$. (4.2) implies that

$$A(\nu,\nu')(\frac{m}{2}\beta)=\frac{\Gamma(2-\nu(Z')+m)}{\Gamma(\nu(Z')+m)}.$$

Theorem 4.2. *There exists a complementary series only if $n=1$ and $l=0$, in which case a unitary structure on $\pi_{l,\nu}$ is given, for each K-type $\frac{m}{2}\beta$, by*

$$\frac{\Gamma(2-\nu(Z')+m)}{\Gamma(\nu(Z')+m)}\int_K |f(k)|^2dk$$

for $f\in L^2(K)$ of K-type $\frac{m}{2}\beta$, and $0<\nu(Z')<2$.

It seems to us that the explicit unitary norm of the Stein complementary series $SL(2N,\mathbb{C})$ in the compact picture has not been found before, even in the case $N=1$.

Acknowledgement

We are grateful to Michael Cowling for helpful discussions. We would also like to thank G. Heckman for explaining the Heckman-Opdam theory of hypergeometric functions and multi-variable Jacobi polynomials. We thank the Australian Research Council for its support.

References

[DOZ] A. Dooley, B. Ørsted and G. Zhang, *Relative discrete series of line bundles over bounded symmetric domains*, Ann. Inst. Fourier **46** (1996), 1011–1026. CMP 97:03

[Hec] G. J. Heckman, *Root systems and hypergeometric functions II*, Comp. Math. **64** (1987), 352-373. MR **89b:**58192b

[HO] G. J. Heckman and E. M. Opdam, *Root systems and hypergeometric functions I*, Comp. Math. **64** (1987), 329-352. MR **89b:**58192a

[He] S. Helgason, *Groups and Geometric Analysis*, Academic Press, London, 1984. MR **86c:**22017

[HT] R. Howe and E. Tan, *Homogeneous Functions on Light Cones: the Infinitesimal Structure of Some Degenerate Principal Series Representations*, Bull. Amer. Math. Soc. **28** (1993), 1-74. MR **93j:**22027

[Jo] K. D. Johnson, *Degenerate principal series and compact groups*, Math. Ann. **287** (1991), 703-718. MR **91h:**22028

[Kn] A. W. Knapp, *Representation Theory of Semisimple Groups*, Princeton University Press, Princeton, New Jersey, 1986. MR **87j:**22022

[KW] A. Korányi and J. Wolf, *Realization of Hermitian Symmetric spaces as Generalized Half planes*, Ann. of Math **81** (1965), 265-268. MR **30:**4980

[KS] B. Kostant and S. Sahi, *Jordan algebras and Capelli Identities*, Invent. Math. **112** (1993), 657-664. MR **94b:**17054

[LP] H. Liu and L. Peng, *Weighted Plancherel formula. Irreducible unitary representations and eigenspace representations*, Math. Scand. **72** (1993), 99-119. MR **94f:**22021

[L] O. Loos, *Bounded Symmetric Domains and Jordan Pairs*, University of California, Irvine, 1977.

[OZ] B. Ørsted and G. Zhang, *Generalized principal series representations and tube domains*, Duke Math. J. **78** (1995), 335-357. MR **96c:**22015

[PZ] J. Peetre and G. Zhang, *Harmonic Analysis on Quantized Riemann Sphere*, Int. J. Math. & Math. Sci. **16** (1993), 225-244. MR **94b:**47034

[Sch] H. Schlichtkrull, *One-dimensional K-types in finite dimensional representations of semisimple Lie groups: A generalization of Helgason's theorem*, Math. Scand. **54** (1994), 279-294. MR **85i:**22014

[Shi] N. Shimeno, *The Plancherel formula for spherical functions with one-dimensional K-type on a simply connected simple Lie group of Hermitian type*, J. Funct. Anal. **121** (1994), 331-388. MR **95b:**22036

[St] E. M. Stein, *Analysis in matrix spaces and some new representations of $SL(N, \mathbb{C})$*, Ann. of Math. **86** (1967), 461-490. MR **36:**2749

[Vo1] D. Vogan, *The unitary dual of $GL(n)$ over an archimedean field*, Invent. Math. **83** (1983), 449-505. MR **87i:**22042

[Vo2] D. Vogan, *Unitary Representations of Reductive Lie Groups*, Annals of Mathematics Studies, Number 118, Princeton University Press, Princeton, New Jersey, 1978.

[V1] L. Vretare, *Elementary spherical functions on symmetric spaces*, Math. Scand. **39** (1976), 343-358. MR **56:**6289

[V2] L. Vretare, *Recurrence formulas for zonal polynomials*, Math. Z. **188** (1985), 419-425. MR **86j:**22012

[Zh1] G. Zhang, *Some recurrence formula for spherical polynomials on tube domains*, Trans. Amer. Math. Soc. **347** (1995), 1725-1734. MR **95h:**22018

[Zh2] G. Zhang, *Jordan algebras and generalized principal series representations*, to appear, Math. Ann. (1995). CMP 95:16

[Zh3] G. Zhang, *A weighted Plancherel formula. The case of the ball*, Studia Math. **102** (1992), 103-120. MR **94e:**43009

School of Mathematics, University of New South Wales, Kensington, New South Wales 2033, Australia

E-mail address: `tony@solution.maths.unsw.edu.au`

Matematiska institutionen, Högskolan i Karlstad, S- 650 09 Karlstad, Sweden

E-mail address: `genkai.zhang@hks.se`