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ON THE IDEAL-TRIANGULARIZABILITY OF POSITIVE OPERATORS ON BANACH LATTICES

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ABSTRACT. There are some known results that guarantee the existence of a nontrivial closed invariant ideal for a quasinilpotent positive operator on an *AM*-space with unit or a Banach lattice whose positive cone contains an extreme ray. Some recent results also guarantee the existence of such ideals for certain positive operators, e.g. a compact quasinilpotent positive operator, on an arbitrary Banach lattice. The main object of this article is to use these results in constructing a maximal closed ideal chain, each of whose members is invariant under a certain collection of operators that are related to compact positive operators, or to quasinilpotent positive operators.

1. PRELIMINARY DEFINITIONS & LEMMAS

Throughout the article we will employ the terminology of the books [3] and [10] and assume familiarity with basic definitions and results of these books. We shall also adhere to the following conventions and definitions.

Unless otherwise stated $\mathbf{N}$ is the set of all natural numbers, X is a general real or complex Banach space, $\mathcal{B}(X)$ is the collection of all bounded linear operators on X, E is a real Banach lattice, $E_+ = \{x \in E : x \geq 0\}$ is the *positive cone* of E, and the topology on $\mathcal{B}(X)$ is the norm topology.

By a *closed invariant ideal* we mean a closed ideal that is taken into itself by the given operator. A subset Γ of $\mathcal{B}(E)$ is called *decomposable* if there exists a nontrivial closed ideal that is invariant for all operators in Γ, otherwise Γ is called *indecomposable*. The collection of all closed invariant ideals (subspaces) of a subset Γ of $\mathcal{B}(E)$ ($\mathcal{B}(X)$) is denoted by $\mathrm{Ilat}(\Gamma)$ ($\mathrm{Lat}(\Gamma)$). If $\Gamma = \{T\}$, where $T \in \mathcal{B}(X)$, we simply use T instead of Γ.

A closed ideal of E is *p-hyperinvariant* for a positive operator T on E if it is invariant under every positive operator S on E which commutes with T. The *positive commutant* of a positive operator on E, denoted by $\{T\}'_+$, is the collection of all positive operators S on E which commutes with T.

A ray $\mathcal{R}_{x_0} = \{\lambda x_0 : \lambda \geq 0\}$, $0 \neq x_0 \in E_+$, is an *extreme ray* for the positive cone E_+ if $x \in \mathcal{R}_{x_0}, x = y + z$, and $y, z \in E_+$ imply $y, z \in \mathcal{R}_{x_0}$. The elements defining extreme rays are also well-known as *atoms*. An *atom* is a positive vector x such that $0 \leq y \leq x$ implies $y = \lambda x$ for some $\lambda \geq 0$. A Banach lattice E is called a *QE-space* if for each pair I, J of closed ideals of E, with $I \subseteq J$, the positive

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cone $(J/I)_+$ of J/I contains an extreme ray. Examples of QE-spaces are discussed in section 4. An example of a Banach lattice which is not a QE-space is given in section 5.

An operator T is said to be *quasinilpotent at* x_0 if $\lim_{n\to 0} \|T^n x_0\|^{1/n} = 0$. The notion of quasinilpotence at a vector was introduced first in [2], where it was shown that it plays an important role for the *invariant subspace problem*.

If I is an ideal of E, then the following proposition (which we use without notice in the sequel) holds for the quotient vector space E/I. (See [12, Propositions II.2.6 and II.5.4].)

Proposition 1.1. *For any ideal I of E, E/I is a vector lattice under the finest ordering of E/I for which the canonical map of E onto E/I is positive. If I is closed, then E/I is a Banach lattice.*

If $T \in \mathcal{B}(X)$ and if $M \in \mathrm{Lat}(T)$, then the *compression* of T to X/M, denoted by $\widehat{T}$ and defined by $\widehat{T}(x+M) = Tx + M$, is a well defined operator on X/M. In this section we state a number of lemmas concerning T. We omit their proofs as they are well known.

Lemma 1.2. *If $T \in \mathcal{B}(E)$ is positive and if $I \in \mathrm{Ilat}(T)$, then $\widehat{T}$ is also a positive operator on E/I.*

Lemma 1.3. *Suppose $T \in \mathcal{B}(X)$ and $M \in \mathrm{Lat}(T)$.*

(a) *If T is quasinilpotent, then $\widehat{T}$ is a quasinilpotent operator on X/M.*

(b) *If T is compact, then $\widehat{T}$ is a compact operator on X/M.*

(c) *If T is weakly compact, then $\widehat{T}$ is a weakly compact operator on X/M.*

Lemma 1.4. *Suppose $S, T \in \mathcal{B}(E)$ and $I \in \mathrm{Ilat}(\{S,T\})$. Let $\widehat{S}$ and $\widehat{T}$ be the compression of S and T to E/I, respectively.*

(a) *If T is a positive operator that dominates S, then $\widehat{T}$ dominates $\widehat{S}$.*

(b) *If $S \leq T$, then $\widehat{S} \leq \widehat{T}$.*

Lemma 1.5. *If I is a closed ideal of an AL- or AM-space E, then I and E/I are AL- or AM-spaces, respectively.*

Lemma 1.6. *If e is the unit of an AM-space E and I is a closed ideal of E, then $e + I$ is the unit of E/I.*

2. Decomposability theorems

We first recall a number of known results concerning decomposability of positive operators. Here by a "positive operator" we mean "a nonzero positive operator".

Theorem 2.1 ([12, Proposition V.6.1])**.** *If E is a closed ideal of an AM-space with unit or if E is a Banach lattice whose positive cone contains an extreme ray and if $T \in \mathcal{B}(E)$ is a quasinilpotent positive operator, then T is decomposable.*

Theorem 2.2 ([9, Proposition 2])**.** *Every compact, quasinilpotent, positive operator on a Banach lattice is decomposable.*

Theorem 2.3 ([1, Theorem 4.3])**.** *Every compact, quasinilpotent, positive operator on a Banach lattice has a nontrivial p-hyperinvariant closed ideal.*

Corollary 2.4. *Suppose $T \in \mathcal{B}(E)$ is a weakly compact quasinilpotent positive operator and E is an AL- or AM-space. Then T^2 has a nontrivial p-hyperinvariant closed ideal. In particular, T has a nontrivial p-hyperinvariant closed ideal.*

Proof. By [5, Corollary 5.19.9], T^2 is a compact operator. The results now follow from Theorem 2.3 and the fact that $\{T\}' \subseteq \{T^2\}'$. □

Corollary 2.5. (a) *Suppose Γ is a commutative collection of positive operators on E and let T be a quasinilpotent positive operator. Then in each of the following cases Γ is decomposable.*

(a) *T is compact and $T \in \Gamma$.*

(b) *T is dominated by a compact operator and $T^3 \in \Gamma$.*

(c) *E is an AM- or AL-space, T is weakly compact, and $T^2 \in \Gamma$.*

(d) *E is an AM- or AL-space, T is dominated by a weakly compact operator, and $T^4 \in \Gamma$.*

Proof. (a) Since $\Gamma \subseteq \{T\}'_+$, Theorem 2.3 gives the desired result.

(b) By [5, Theorem 5.16.13], T^3 is compact and hence Γ is decomposable by (a).

(c) As in the proof of Corollary 2.4, T^2 is compact. Now apply (a).

(d) By [5, Theorem 5.17.11], T^2 is weakly compact and hence Γ is decomposable by (c). □

Our reading of [1] and [9] revealed that there is an extension of Theorem 2.3, whose proof is a slight modification of the proofs given in [9] and [1] for Theorems 2.2 and 2.3, respectively.

Theorem 2.6. *Let $T \in \mathcal{B}(E)$ be a nonzero compact positive operator. If T is quasinilpotent at some $x_0 > 0$ in E, then T has a nontrivial p-hyperinvariant closed ideal.*

Proof. Since the *null ideal* $N_T = \{x \in E : T(|x|) = 0\}$ is a p-hyperinvariant closed ideal for T, we are done if $Tx_0 = 0$. If $Tx_0 \neq 0$ we proceed as follows:

As in the proof of [1, Theorem 4.3], $\overline{E_u} = F$, where

$$F = \overline{\{x \in E : \exists y \geq 0 \text{ such that } |x| \leq Ty\}}\,,$$

$u = \sum_{n=1}^{\infty} \frac{|x_n|}{2^n \|x_n\|} > 0$, and $\{x_n\}$ is a norm dense sequence in $T(E)$. Hence F is a nontrivial p-hyperinvariant closed ideal for T, if E does not contain a quasi-interior point. Otherwise, as in the second part of [9, Proposition 2],

For each $f > 0$ the closure $\overline{\mathcal{T}[f]}$ is a nonzero closed ideal which is invariant under T, where $\mathcal{T}[f] = \{Sf : S \in \mathcal{T}\}, \mathcal{T} = \{S_1 - S_2 : S_1, S_2 \in \mathcal{T}^+\}$, and $\mathcal{T}^+ = \{S \in \mathcal{B}(E) : 0 \leq S \leq R$ for some $R \in \{T\}'_+\}$.

By the last part of [1, Theorem 4.3], $\overline{\mathcal{T}[f]}$ is a p-hyperinvariant closed ideal for T; therefore we are done if we show that there exists $f \neq 0$ in E such that $\overline{\mathcal{T}[f]} \neq E$. Suppose, on the contrary, that $\overline{\mathcal{T}[f]} = E$ for all $f \neq 0$, in E. Since x_0 and Tx_0 are not equal to zero, we can choose an open ball $\mathcal{U}$, with center x_0, such that $0 \neq \overline{T(\mathcal{U})}$ and $0 \neq \overline{\mathcal{U}}$. Proceed as in the final step of the proof of [9, Proposition 2] to find a natural number n, a sequence $\{j_m\}_{m=1}^{\infty}$ in $\{1, 2, \ldots, n\}$, and a sequence $\{S_j\}_{j=1}^{n}$ in $\mathcal{T}$ such that $g_m = S_{j_m} T S_{j_{m-1}} T \cdots S_{j_1} T x_0 \in \mathcal{U}$ for all m, and show that $\|g_m\| \leq (2C)^m \|T^m x_0\|$, where $C = \max\{\|R_j^{(i)}\| : j = 1, \ldots, n; i = 1, 2\}$. Since T is quasinilpotent at x_0, $\|g_m\| \to 0$ and hence $0 \in \overline{\mathcal{U}}$, contradicting the choice of $\mathcal{U}$. □

Remark. It is clear that similar corollaries to those of Theorem 2.3 can be obtained from Theorem 2.6 if we replace quasinilpotency by quasinilpotency at some $x_0 > 0$ for which $Tx_0 > 0$.

To introduce other decomposability theorems we need to state a simple fact whose proof is omitted.

Lemma 2.7. *Suppose* $\Gamma \subseteq \mathcal{B}(E), T \in \mathcal{B}(E), T$ *is a positive operator, and* $I \in \mathrm{Ilat}(T)$. *Then* $I \in \mathrm{Ilat}(\Gamma)$ *if either* T *dominates all of the operators in* Γ *or if all elements of* Γ *are positive and* $S \leq T$ *for all* $S \in \Gamma$.

By this Lemma and all the results preceding it, we can easily obtain other decomposability results. Here are some examples:

Corollary 2.8. *Suppose* $\Gamma \subseteq \mathcal{B}(E), T$ *is a quasinilpotent positive operator on* E *and either* (a) T *dominates all members of* Γ *or* (b) *all elements of* Γ *are positive and* T *majorizes* Γ. *Then in each of the following cases* Γ *is decomposable*; (i) E *is a closed ideal of an AM-space,* (ii) E *is a Banach lattice whose positive cone contains an extreme ray,* (iii) E *is an AL-space and* T *is weakly compact,* (iv) E *is any Banach space and* T *is compact.*

3. Ideal chains and ideal triangularizability

The concept of *triangularizability* of a collection of operators on a finite or infinite dimensional Banach space has been studied by many authors, e.g., [6], [7], [8], and [10]. Recall that the collection Γ of operators, on a Banach space, is *triangularizable* if there is a maximal subspace chain each of whose members is invariant under all the operators in Γ.

It is the purpose of this section to introduce a Banach lattice version of this concept and a few of its related results.

Let $A(E)$ denote the collection of all closed ideals of E and partially order $A(E)$ by the inclusion relation "$\subseteq$". A totally ordered subset $\mathcal{F}$ of $A(E)$ will be called an *ideal chain*. If each element of $\mathcal{F}$ is invariant under a collection of operators Γ on E, we shall call $\mathcal{F}$ an *invariant ideal chain*. A trivial example of an invariant ideal chain is $\{\{0\}, E\}$. The existence of nontrivial invariant ideal chains for several classes of operators on Banach lattices can be deduced from some of the theorems in section 2. A similar argument to the one given in [11, Section 4.3], shows that if Υ, the class of all ideal chains, is partially ordered by the inclusion relation then by an application of Zorn's Lemma one can show the existence of *maximal* (*invariant*) *ideal chains* in Υ.

Definition 3.1. The collection Γ of operators on a Banach lattice E is *ideal-triangularizable* if there is a maximal ideal chain each of whose members is invariant under all the operators in Γ; such an ideal chain will be said to be triangularizing for Γ. If $\Gamma = \{T\}$, we simply say that T is ideal-triangularizable.

Let $\mathcal{F}$ be an ideal chain and let $\mathcal{F}_0$ be a subfamily of $\mathcal{F}$. It is clear that $\Theta_0 = \bigcap\{I : I \in \mathcal{F}_0\}$ is a closed ideal. Since $\mathcal{F}_0$ is totally ordered by inclusion the set $\Omega_0 = \bigcup\{I : I \in \mathcal{F}_0\}$ is a linear manifold of E. Now suppose $x \in \Omega_0$ and $y \in E$ are such that $|y| \leq |x|$. Let $I \in \mathcal{F}_0$ be such that $x \in I$. Since I is an ideal, $y \in I$ and hence $y \in \Omega_0$. Thus Ω_0 is an ideal of E and hence the norm closure of Ω_0, i.e. $\overline{\Omega_0}$, is a closed ideal of E.

Definition 3.2. Let $\mathcal{F}$ be an ideal chain. We call $\mathcal{F}$ a *simple ideal chain* if it satisfies: (1) $\{0\} \in \mathcal{F}$, and $E \in \mathcal{F}$, (2) if $\mathcal{F}_0$ is a subfamily of $\mathcal{F}$, then the closed ideals Θ_0 and $\overline{\Omega_0}$ are in $\mathcal{F}$, (3) for each J in $\mathcal{F}$, the quotient J/J_{-} is at most

1-dimensional, where $J_- = [\bigcup\{I : I \in \mathcal{F}, I \neq J, \text{ and } I \subset J\}]$ if there are closed ideals I in $\mathcal{F}$ which are properly contained in J and $J_- = \{0\}$ otherwise.

With exactly the same proof of [11, Lemma 4.3.1], one can show:

Lemma 3.3. *Each simple ideal chain is maximal.*

Definition 3.4. A subset Γ of $\mathcal{B}(E)$ is said to be *compressionally decomposable* if for any $I, J \in \text{Ilat}(\Gamma)$, with $I \subseteq J$ and $\dim(J/I) \geq 2$, the compression $\widehat{\Gamma}$ of Γ to J/I is decomposable.

Proposition 3.5. *Suppose E is a closed ideal of an AM-space F with unit f. If $T \in \mathcal{B}(E)$ is a quasinilpotent positive operator, then T is compressionally decomposable.*

Proof. By Theorem 2.1 T is decomposable. Let $I, J \in \text{Ilat}(T)$, with $I \subseteq J$ and $\dim(J/I) \geq 2$. Since E is an AM-space, J is an AM-space. Since I is a closed ideal of both J and F, J/I is a closed ideal of F/I, J/I and F/I are both AM-spaces by Lemma 1.5, and $f + I$ is a unit for F/I by Lemma 1.6.

Since T is a quasinilpotent positive operator on E, T is a quasinilpotent positive operator on J. Hence $\widehat{T}$ is a quasinilpotent positive operator on J/I by Lemma 1.2 and Lemma 1.3(a). Therefore, Theorem 2.1 implies that $\widehat{T}$ is decomposable. ☐

Proposition 3.6. *Suppose E is a QE-space and T is a quasinilpotent positive operator on E. Then T is compressionally decomposable.*

Proof. Since E is a QE-space, $E_+ = (E/\{0\})_+$ contains an extreme ray and hence T is decomposable by Theorem 2.1. Let $I, J \in \text{Ilat}(T)$, with $I \subseteq J$ and $\dim(J/I) \geq 2$. Since E is a QE-space, $(J/I)_+$ contains an extreme ray. Since $\widehat{T}$ is a quasinilpotent positive operator on J/I by Lemmas 1.2 and 1.3(a), $\widehat{T}$ is decomposable by Theorem 2.1. ☐

Proposition 3.7. *If $K \in \mathcal{B}(E)$ is a compact, quasinilpotent, positive operator, then K and $\Gamma = \{K\}'_+$ are compressionally decomposable.*

Proof. By Theorem 2.3, K and Γ are decomposable. Let $I, J \in \text{Ilat}(K)$ or $\text{Ilat}(\Gamma)$, with $I \subseteq J$ and $\dim(J/I) \geq 2$. We know that $\widehat{K}$ is a compact quasinilpotent positive operator on J/I by Lemma 1.2, Lemma 1.3(a), and Lemma 1.3(b). Hence, by Theorem 2.3, $\widehat{K}$ and the compression $\widehat{\Gamma}$, of Γ to J/I, are decomposable, as $\widehat{\Gamma} \subseteq \{\widehat{K}\}'$. ☐

Proposition 3.8. *Suppose $K \in \mathcal{B}(E)$ is a weakly compact quasinilpotent positive operator and E is an AL- or AM-space. Then K and $\{K^2\}'_+$ are compressionally decomposable. In particular $\{K\}'_+$ is compressionally decomposable.*

Proof. Apply Corollary 2.4, Lemma 1.2, Lemma 1.3 and Lemma 1.5. ☐

Proposition 3.9. *If $T \in \mathcal{B}(E)$ is a nonzero quasinilpotent positive operator and if there exists a nonzero operator $S \in \{T\}'_+$ that dominates a nonzero compact operator $K \in \mathcal{B}(E)$, then $\{S, T\}$ is compressionally decomposable.*

Proof. Since T is a quasinilpotent positive operator, there exists $x_0 > 0$ in E such that T is quasinilpotent at x_0, and hence, by [3, Theorem 7.1], $\{S, T\}$ is decomposable. Suppose $I, J \in \text{Ilat}(\{S, T\})$. It is easy to see that $I, J \in \text{Ilat}(K)$ and, by Lemma 1.4, the compression $\widehat{S}$ of S to J/I dominates the compression $\widehat{K}$

of K to J/I. Now apply the appropriate lemmas of section 1 and [3, Theorem 7.1] to prove $\{\widehat{S}, \widehat{T}\}$ is decomposable, if $\dim(J/I) \geq 2$. □

The following proposition and its corollaries are easy consequences of the above results. We omit their easy proofs.

Proposition 3.10. *Suppose $\Gamma \subseteq \mathcal{B}(E)$ and $T \in \Gamma$ is a compressionally decomposable positive operator. Then Γ is compressionally decomposable if either T dominates all of the elements of Γ or if all of the elements of Γ are positive operators and if T majorizes Γ.*

Corollary 3.11. *Suppose $\Gamma \subseteq \mathcal{B}(E), T \in \Gamma$ is a quasinilpotent positive operator, and either* (a) *T dominates all elements of Γ or* (b) *all elements of Γ are positive and T majorizes Γ. Then Γ is compressionally decomposable in each of the following cases*: (i) *E is a closed ideal of an AM-space,* (ii) *E is a QE-space,* (iii) *E is an AL-space and T is weakly compact,* (iv) *E is any Banach lattice and T is compact.*

Lemma 3.12. *Suppose $\Gamma \subseteq \mathcal{B}(E)$ is compressionally decomposable. Then each maximal invariant ideal chain for Γ is simple.*

Proof. Suppose that $\mathcal{F}$ is a maximal invariant ideal chain for Γ. By a similar proof to the one given in the first part of the proof of [11, Lemma 4.3.2], we can show that $\mathcal{F}$ has the first two conditions of Definition 3.2.

To verify the third, suppose $\dim(J/J_-) \geq 2$, for some J in $\mathcal{F}$. Since Γ is compressionally decomposable, there exists a nontrivial closed ideal I_0 of J/J_- such that $\widehat{T}(I_0) \subseteq I_0$ for all $\widehat{T} \in \widehat{\Gamma}$.

If we define $I = \{x \in J : x + J_- \in I_0\}$, then, as the canonical map $\pi : E \to E/I$ is a lattice homomorphism, I is a closed ideal of E. It is also easy to verify that $T(I) \subseteq I$ for all $T \in \Gamma$ and $J_- \subsetneqq I \subsetneqq J$.

As in the last part of the proof of [11, Lemma 4.3.2], we can show that $I \notin \mathcal{F}$ and $\mathcal{F} \cup \{I\}$ is totally ordered by inclusion. Hence $\mathcal{F} \cup \{I\}$ is an invariant ideal chain for Γ that properly contains $\mathcal{F}$, which contradicts the maximality of $\mathcal{F}$. This contradiction implies that $\dim(J/J_-)$ is at most one and hence $\mathcal{F}$ is a simple ideal chain. □

Theorem 3.13. *Let $\Gamma \subseteq \mathcal{B}(E)$ be compressionally decomposable. Then there is a simple ideal chain $\mathcal{F}$ such that each $I \in \mathcal{F}$ is invariant under each $T \in \Gamma$.*

Proof. It is sufficient to take for $\mathcal{F}$ any maximal invariant ideal chain, by Lemma 3.12. □

By using Propositions 3.5–3.10, Corollary 3.11, and Theorem 3.13, we are now able to derive some ideal-triangularizability results. We present one and leave the rest as they can be obtained similarly.

Theorem 3.14. *Suppose E is a closed ideal of an AM-space with unit, in particular suppose $E = C_0(X)$, where X is a locally compact Hausdorff space. If T is a quasinilpotent positive operator on E, then T is ideal-triangularizable.*

4. CB-spaces and ideal-triangularizability

If X is a Banach space with a *Schauder basis* $\{x_n\}_{n=1}^{\infty}$, then this basis gives rise to a natural closed cone C defined by $C = \{x = \sum_{n=1}^{\infty} \alpha_n x_n : \alpha_n \geq 0 \ \forall n\}$. With this cone, X will be an *ordered Banach space*. Now if the basis is unconditional, then by using [13, Theorem 16.1] we can check that this ordered Banach space is indeed a Banach lattice and for every $x \in X$, with $x = \sum_{n=1}^{\infty} \alpha_n x_n$, $|x| = \sum_{n=1}^{\infty} |\alpha_n| x_n$. Conversely, if a Banach lattice E has a Schauder basis and if the positive cone of E_+ is compatible with the cone generated by this basis, then this basis is unconditional. Such Banach lattices are called *CB-spaces*.

Recently in [4], the authors established an invariant subspace theorem for the positive operators acting on a Banach space X with a basis. In this section it is shown that in that theorem "subspace" can be replaced by "closed ideal" if the basis of X is unconditional.

In what follows the sequence $\{x_n\}_{n=1}^{\infty}$ is an unconditional basis for the given CB-space.

Definition 4.1. Let I be an ideal of a CB-space E. We say that x_i *participates* in I if there exists $x \in I$ with $x = \sum_{n=1}^{\infty} \alpha_n x_n$ such that $\alpha_i \neq 0$.

Lemma 4.2. *Suppose I is an ideal of a CB-space E and $\mathcal{P} = \{x_i : x_i$ participates in $I\}$. Then $\mathcal{P} \subset I$, and hence $I = \{x \in E : x = \sum_{x_i \in \mathcal{P}} \alpha_i x_k\}$.*

Proof. Suppose $x_i \in \mathcal{P}$; then there exists $x = \sum_{n=1}^{\infty} \alpha_n x_n$ in I such that $\alpha_i \neq 0$. Since $|x| \subset I$ and since $|x| = \sum_{n=1}^{\infty} |\alpha_n| x_n$, the relation $|\alpha_i| x_i \leq \sum_{n=1}^{\infty} |\alpha_n| x_n$ implies $|\alpha_i| x_i \in I$. Hence $x_i \in I$ as $\alpha_i \neq 0$. □

Lemma 4.3. *Suppose E is a CB-space. Then I is a closed ideal of E if and only if there exists a subset S_I of $\mathbf{N}$ such that*

$$I = \left\{ x = \sum_{n=1}^{\infty} \alpha_n x_n \in E : \alpha_m = 0 \ \forall m \notin S_I \right\}.$$

Proof. Suppose I is a closed ideal of E. If $I = \{0\}$ take $S_I = \varnothing$, and if $I = E$ take $S_I = \mathbf{N}$. Suppose $I \neq \{0\}$ and $I \neq E$. If such S_I does not exist, then for each $i \in \mathbf{N}$ there exists $x \in I$ with $x = \sum_{n=1}^{\infty} \alpha_n x_n$ such that $\alpha_i \neq 0$. This x_i participates in I for all $i \in \mathbf{N}$ and hence I contains all elements of the basis $\{x_n\}_{n=1}^{\infty}$ by Lemma 4.2. Therefore, $I = E$ as I is closed, a contradiction.

Conversely, suppose S_I is a subset of $\mathbf{N}$ for which I is the collection of all $x = \sum_{n=1}^{\infty} \alpha_n x_n$ in E with $\alpha_m = 0$ for all $m \notin S_I$. It is easy to verify that I is an ideal of E. Now let $\{y_k\}_{k=1}^{\infty}$ be a sequence in I that converges to y in E. Suppose that, for each k, $\{\beta_{kn}\}_{n \in S_I}$ is a sequence of scalars such that $y_k = \sum_{n \in S_I} \beta_{k,n} x_n$. Suppose also that $\{\beta_n\}_{n=1}^{\infty}$ is a sequence of scalars such that $y = \sum_{n=1}^{\infty} \beta_n x_n$. Since for each k, $\|y_k - y\| = \|\sum_{n=1}^{\infty} \gamma_{k,n} x_n\|$, where $\gamma_{k,n} = \beta_{k,n} - \beta_n$ for $n \in S_I$ and $\gamma_{k,n} = -\beta_n$ for $n \notin S_I$, and since $\|y_k - y\| \to 0$, we should have $\lim_{k\to\infty} \beta_{k,n} = \beta_n$ whenever $n \in S_I$ and $\lim_{k\to\infty} \beta_n = 0$ whenever $n \notin S_I$, as $\lim_{k\to\infty} \gamma_{k,n} = 0$ for all $n \in \mathbf{N}$. Hence $\beta_n = 0$ for all $n \notin S_I$, which means $y \in I$ and hence I is closed. □

Corollary 4.4. *For each closed ideal $I \neq \{0\}$, of a CB-space E, there exists $i \in \mathbf{N}$ such that $x_i \in I$.*

Proof. If $I = E$ there is nothing to prove. If $I \neq E$, let S_I be as in Lemma 4.3. Since S_I is not empty, there exists $i \in S_I$. For this i we have $x_i \in I$. □

Corollary 4.5. *The positive cone of each closed ideal I of a CB-space E contains an extreme ray.*

Proof. If $I = \{0\}$ there is nothing to prove. Suppose $I \neq \{0\}$ and let S_I be as in Lemma 4.3. By Corollary 4.4, let $i \in \mathbf{N}$ be such that $x_i \in I$. Suppose $\lambda \geq 0$ and $x, y \in I_+$ are such that $\lambda x_i = x + y$, and suppose $\{\alpha_n\}_{n\in S_I}, \{\beta_n\}_{n\in S_I}$ are two sequences of nonnegative scalars such that $x = \sum_{n\in S_I} \alpha_n x_n$ and $y = \sum_{n\in S_I} \beta_n x_n$. Then, by the properties of a basis we should have $\alpha_n+\beta_n = 0$ for all $n \in S_I \setminus \{i\}$ and $\alpha_i+\beta_i = \lambda$. Hence $\alpha_n = \beta_n = 0$ for all $n \in S_I \setminus \{i\}$. This means $x, y \in \{\mu x_i : \mu \geq 0\}$ and hence I_+ contains an extreme ray. □

Corollary 4.6. *Suppose I, J, with $I \subseteq J$ are two closed ideals of a CB-space E. Then the positive cone $(J/I)_+$ of the quotient space J/I contains an extreme ray. Furthermore, each CB-space is a QE-space.*

Proof. If either $J = \{0\}$ or $J = I$ there is nothing to prove, hence suppose $J \neq I$ and let S_I and S_J be as in Lemma 4.3. Clearly, $S_i \subsetneqq S_J$. Let $j \in S_J$ be such that $x_j \notin I$. By Corollary 4.5, x_j generates an extreme ray for J_+.

Let $x, y \in J$ be such that $x + I, y + I \in (J/I)_+$. (Without loss of any generality we can assume that $x, y \in J_+$.) Suppose the sequences of positive scalars $\{\alpha_n\}_{n\in S_J}$ and $\{\beta_n\}_{n\in S_J}$ are such that $x = \sum_{n\in S_J} \alpha_n x_n$ and $y = \sum_{n\in S_J} \beta_n x_n$. If $\lambda \geq 0$ is such that $\lambda(x_j + I) = (x + I) + (y + I)$, then there exists $z \in I$ such that $\lambda x_j - x - y = z$. Hence

$$(\lambda - \alpha_j - \beta_j)x_j + \sum_{n\in S} -(\alpha_n + \beta_n)x_n + \sum_{n\in S_I} -(\alpha_n + \beta_n + \gamma_n)x_n = 0,$$

where $S = S_J \setminus (S_I \cup \{j\})$ and $\{\gamma_n\}_{n\in S_I}$ is a sequence of scalars such that $z = \sum_{n\in S_I} \gamma_n x_n$. Thus, $\lambda = \alpha_j + \beta_j, \alpha_n + \beta_n = 0$ for all $n \in S$, and $\alpha_n + \beta_n + \gamma_n = 0$ for all $n \in S_I$ and hence $\alpha_n = \beta_n = 0$ for all $n \in S$. Therefore, $x + I = \alpha_j(x_j + I)$ and $y+I = \beta_j(x_j+I)$, which means x_j+I generates an extreme ray for $(J/I)_+$. □

Corollaries 4.6 and 3.11-(ii), and the comment preceding Theorem 3.14, now imply the main result of this section concerning CB-spaces and, in particular, the Banach lattices c_0 and l_p, where $l \leq p < \infty$. In fact:

Theorem 4.7. *Suppose E is a CB-space, T is a quasinilpotent positive operator on E, and $\Gamma \subseteq \mathcal{B}(E)$. Then T is ideal-triangularizable. If $T \in \Gamma$ and T dominates all elements of Γ, then Γ is also ideal-triangularizable. Furthermore, if all elements of Γ are positive and if T majorizes Γ, then Γ is ideal-triangularizable.*

5. Examples, questions & comments

Example 5.1. Let $E = L^p[0, 1]$, $1 \leq p < \infty$, or $E = C[0, 1]$, and let V be the Volterra integral operator on E, i.e., the indefinite integral $Vf(t) = \int_0^t f(x)\,dx$. It is known that V is a compact quasinilpotent operator on E and it is obvious that V is a positive operator in both cases.

For each $s \in [0, 1]$, let $I_s = \{f \in E : f = 0 \text{ a.e. on } [0, s]\}$ if $E = L^p[0, 1]$, and let $I_s = \{f \in E : f = 0 \text{ on } [0, s]\}$ if $E = C[0, 1]$. In either case it is clear that the I_s form closed ideals of E which are invariant under V. In a similar discussion to that of [11, Example 4.2.12], it can also be shown that, in both cases, the family $\mathcal{F} = \{I_s : 0 \leq s \leq 1\}$ is a simple chain, and hence $\mathcal{F}$ is a triangularizing ideal chain for V.

Example 5.2. Let $K = [0,1]$ and let $E = C(K)$. For each nonzero $f \in E_+$, let α and β be two positive numbers such that $[\alpha, \beta] \subset [0,1]$ and $f(t) > 0$ for all $t \in [\alpha, \beta]$. Let g be a continuous function on $[\alpha, \beta]$ such that $g(t) > 0$ for all $t \in [\alpha, \beta]$, $f(\alpha) = g(\alpha)$, $f(\beta) = g(\beta)$, and $f(t) > g(t)$ for all $T \in (\alpha, \beta)$. Define h on $[\alpha, \beta]$ by $h(t) = 2f(t) - g(t)$. Then h is also continuous on $[\alpha, \beta]$, $h(t) > 0$ for all $t \in [\alpha, \beta]$, $f(\alpha) = h(\alpha)$, $f(\beta) = h(\beta)$, and $h(t) > f(t)$ for all $t \in (\alpha, \beta)$.

Now if we define g, h on $[0, \alpha] \cup [\beta, 1]$ by $g(t) = h(t) = f(t)$, then it is easy to show that: $g, h \in E_+$, $f(t) = [g(t) + h(t)]/2$ for all $t \in [0,1]$, and g, h are not a positive multiple of f. This shows that E_+ has no extreme rays, yet every quasinilpotent positive operator on E is decomposable, by Theorem 2.1.

Our results concerning a quasinilpotent positive operator on a CB-space give rise to the following question:

Question 5.3. Is it true that every quasinilpotent positive operator on a Banach lattice, whose positive cone contains an extreme ray, is ideal triangularizable?

Remark 5.4. The procedure in the proof of ideal-triangularizability of a quasinilpotent positive operator, on a closed ideal of an AM-space with unit or on a CB-space, may not work to answer Question 5.3. For example, the Banach lattice $E = R \oplus C[0,1]$ has an extreme ray, but, as we see in Example 5.2, the positive cone I_+ of the closed ideal $I = \{0\} \oplus C[0,1]$ has no extreme rays.

In view of Theorem 7.1 of [3] and the results in [6], [7], [8], and [10] the investigation of answers to the following questions seems to be interesting.

Question 5.5. Let T be a positive operator on a Banach lattice E which is quasinilpotent at an element $x_0 > 0$ in E. Under what conditions does there exist a polynomial p, with positive coefficients, or $S \in \{T\}'_+$ such that either $p(T)$ or S dominates a nonzero compact operator on E?

Question 5.6. Under what conditions is a semigroup $\mathcal{S}$ of quasinilpotent positive operators on E decomposable or ideal-triangularizable, if
(a) E is a closed ideal of an AM-space or
(b) E is a Banach lattice whose positive cone contains an extreme ray or
(c) E is an arbitrary Banach lattice and $\mathcal{S}$ contains some compact operators?

Of course, one can obtain some partial answers, for Question 5.6, by using the results of Sections 2 and 3.

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