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NOTE ON FAITHFUL REPRESENTATIONS AND A LOCAL PROPERTY OF LIE GROUPS

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ABSTRACT. Let G be any analytic group, let T be a maximal toroid of the radical of G, and let S be a maximal semisimple analytic subgroup of G.

If $L = \mathcal{L}(G)$ is the Lie algebra of G, $\operatorname{rad}[L, L]$ is the radical of $[L, L]$, and $\mathcal{Z}(L)$ is the center of L, we show that G has a faithful representation if and only if

(i) $\operatorname{rad}[L, L] \cap \mathcal{Z}(L) \cap \mathcal{L}(T) = (0)$, and

(ii) S has a faithful representation.

A theorem of M. Moskowitz [4, Thm. 2], shows that if L is a finite-dimensional (real) Lie algebra, then all analytic groups with Lie algebra L have faithful representations if and only if (i) $\operatorname{rad}[L, L] \cap \mathcal{Z}(L) = (0)$, and (ii) for some maximal semisimple subalgebra S of L, the simply connected analytic group with Lie algebra S has a faithful representation. So it would be of interest to find a similar criterion for a single analytic group G to have a faithful representation. Such a criterion is given in Theorem 2 below. As a consequence, we obtain Moskowitz' Theorem in Corollary 3. So our criterion in the solvable case says that G has a faithful representation if and only if $[L, L] \cap \mathcal{Z}(L) \cap \mathcal{L}(T) = (0)$ for some maximal toroid T of G where $L = \mathcal{L}(G)$; whereas the well-known criterion in the solvable case is that G has a faithful representation if and only if $[G, G]$ is closed in G and simply connected [2, p. 220]. For the case of semisimple analytic groups, we refer the reader to [2, pp. 199–201].

Our proof uses the notion of nuclei of analytic groups introduced by Hochschild and Mostow. A *nucleus* K of an analytic group G is a closed normal simply connected solvable analytic subgroup of G such that G/K is reductive. An analytic group G has a faithful representation if and only if G has a nucleus; if G has a nucleus K, then $G = K \cdot P$ (semi-direct) for every maximal reductive analytic subgroup P of G [3, Section 2]. Recall that an analytic group is *reductive* if it has a faithful representation and all its representations are semisimple.

If G is an analytic group, $\mathcal{L}(G)$ is its Lie algebra, $\operatorname{rad} G$ is its radical, and $[G, G]$ is its commutator (derived) subgroup. Similarly, if L is a Lie algebra, $\operatorname{rad} L$ is its radical, and $[L, L]$ is its commutator (derived) subalgebra. *All representations of analytic groups are assumed to be analytic and finite dimensional.*

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Lemma 1. *Let G be any analytic group, and let A and B be analytic subgroups of G such that A is normal in G. If $G = AB$ and $A \cap B = (1)$, then A and B are closed in G.*

Proof. Let $G^+ = A \times B$ be the cartesian product of the analytic manifolds of A and B underlying the analytic groups A and B. Then, in addition to its manifold structure, $G^+ = A \times B$ is also an abstract group via the conjugation action of B on A in G. We now show that these structures turn G^+ into an analytic group. Let $f\colon\ A \times B \to A$ be the mapping given by $f(a,b) = bab^{-1}$. Then one can easily check that f is analytic on a neighborhood of $(1_A, 1_B)$ by using the exponential maps in the analytic groups A, B and G. Since A is connected, it follows that f is analytic on $A \times B$ [1, Lemma 3, p. 362].

Hence $G^+ = A \times B$ is an analytic group with the above group and manifold structures. Now let $p\colon\ G^+ = A \times B \to AB = G$ be the mapping given by $p(a,b) = ab$. Then p is a surjective continuous homomorphism between locally compact connected topological groups, so p must be an open map [2, Thm. 2.5, p. 7] or [2, Exercise 1, p. 13]. But p is also bijective since $A \cap B = (1)$. Consequently, $p\colon\ G^+ \to G$ is an isomorphism of analytic groups. Hence A and B are closed in G since they are closed in G^+. □

We shall need the fact that if G has a faithful representation, then its representation radical $N = \mathrm{rad}[G,G]$ is closed in G and simply connected. This is true, for example, because N is contained in every nucleus K of G [3, Section 2] and each K is a closed simply connected solvable analytic subgroup of G. For a direct proof, see the proof of Theorem 1 in [4].

Theorem 2. *Let G be any analytic group with Lie algebra L, let T be a maximal toroid of* $\mathrm{rad}(G)$*, and let S be a maximal semisimple analytic subgroup of G. Then G has a faithful representation if and only if*

(i) $\mathrm{rad}[L,L] \cap \mathcal{Z}(L) \cap \mathcal{L}(T) = (0)$*, and*

(ii) *S has a faithful representation.*

Proof. Let Ad and ad be the adjoint representations of G and $\mathcal{L}(G)$ respectively on the Lie algebra L of G. Since $\mathrm{rad}[L,L]$ acts nilpotently on any representation space of L [2, Thm. 3.2, p. 128], $\mathrm{ad}(\mathrm{rad}[L,L])$ consists of nilpotent elements. Since $\mathrm{ad}(\mathcal{L}(T)) = \mathcal{L}(\mathrm{Ad}(T))$ and T is a toroid, it follows that $\mathrm{ad}(\mathcal{L}(T))$ consists of semisimple elements. Hence $\mathrm{ad}(\mathrm{rad}[L,L] \cap \mathcal{L}(T)) = (0)$. Thus $(\mathrm{rad}[L,L] \cap \mathcal{L}(T)) \subseteq \mathcal{Z}(L)$. Since $\mathrm{rad}[L,L] = \mathcal{L}(N)$ where $N = \mathrm{rad}[G,G]$ [2, Thm. 3.1, p. 138], it follows that $\mathrm{rad}[L,L] \cap \mathcal{Z}(L) \cap \mathcal{L}(T) = (0)$ if and only if $\mathcal{L}(N) \cap \mathcal{L}(T) = (0)$.

So first suppose that G has a faithful representation. Then S has a faithful representation. Moreover, as remarked above, N is a closed simply connected solvable analytic subgroup of G, so N has no non-trivial compact subgroups [2, Thm. 2.3, p. 138]. Hence $\mathcal{L}(N) \cap \mathcal{L}(T) = (0)$.

Conversely, suppose $\mathcal{L}(N) \cap \mathcal{L}(T) = (0)$. Then there exists a subspace $\underline{K}$ of $\mathrm{rad}\,\mathcal{L}(G)$ containing $\mathcal{L}(N)$ such that $\mathrm{rad}\,\mathcal{L}(G) = \underline{K} \oplus \mathcal{L}(T)$. Since $\underline{K}$ contains $\mathcal{L}(N) = \mathrm{rad}[L,L]$ and $\mathrm{rad}[L,L] = [L, \mathrm{rad}\,L]$ [2, Thm. 3.2, p. 128], it follows that $\underline{K}$ is an ideal of $\mathcal{L}(G)$. Hence $\mathrm{rad}\,\mathcal{L}(G) = \underline{K} + \mathcal{L}(T)$ (semi-direct). Thus if K is the analytic subgroup of G corresponding to $\underline{K}$, then K is normal in $G, \mathrm{rad}(G) = K \cdot T$, and the subgroup $K \cap T$ is discrete in the analytic group K. Thus the projection morphism $K \to K/(K \cap T)$ is a covering. Moreover, $K/(K \cap T)$ is homeomorphic to $\mathrm{rad}(G)/T$ which is known to be a simply connected homogeneous space since T

is a maximal toroid of $\mathrm{rad}(G)$ [2, Exercise 1, p. 187]. Hence the covering morphism $K \to K/(K \cap T)$ is a homeomorphism of simply connected homogeneous spaces. Thus $K \cap T = (1)$ and K is simply connected. Since $\mathrm{rad}(G) = K \cdot T$, it follows by Lemma 1 that K is closed in $\mathrm{rad}(G)$. Hence K is a nucleus of $\mathrm{rad}(G)$. Consequently, $\mathrm{rad}(G)$ has a faithful representation [3, Section 2]. Since S has also a faithful representation, it follows that G has a faithful representation [2, Thm. 4.2, p. 221]. This proves Theorem 2. □

Corollary 3 (Thm. 2 of [4])**.** *Let L be a finite-dimensional* (*real*) *Lie algebra. Let S be a maximal semisimple subalgebra of L, and let S^* be the simply connected analytic group with Lie algebra S. Then all analytic groups with Lie algebra L have faithful representations if and only if*

(i) $\mathrm{rad}[L, L] \cap \mathcal{Z}(L) = (0)$*, and*
(ii) S^* *has a faithful representation.*

Proof. Suppose $\mathrm{rad}[L, L] \cap \mathcal{Z}(L) = (0)$, and S^* has a faithful representation. Let G be any analytic group with Lie algebra L, and let S_g be the (maximal) semisimple analytic subgroup of G corresponding to the Lie algebra S. Since S^* has a faithful representation, it follows that S_g also has a faithful representation [4, Cor. 1a]. Hence G has a faithful representation by Theorem 2.

For the converse, we may use the proof in [4, top of p. 197] since it refers only to the fact that $N = \mathrm{rad}[G, G]$ is simply connected whenever G has a faithful representation. □

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