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ON SOLUTIONS OF REAL ANALYTIC EQUATIONS

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Abstract. Analyticity of $\mathcal{C}^\infty$ solutions $y_i = f_i(x), 1 \leq i \leq m$, of systems of real analytic equations $p_j(x,y) = 0, 1 \leq j \leq l$, is studied. Sufficient conditions for $\mathcal{C}^\infty$ and power series solutions to be real analytic are given in terms of iterative Jacobian ideals of the analytic ideal generated by $p_1, p_2, \dots, p_l$. In a special case when the p_i's are independent of x, we prove that if a $\mathcal{C}^\infty$ solution h satisfies the condition $\det\left(\frac{\partial p_i}{p y_j}\right)(h(x)) \not\equiv 0$, then h is necessarily real analytic.

It is well known that a $\mathcal{C}^\infty$ function or a formal power series that satisfies a nonzero real analytic equation is necessarily real analytic. (See Corollary 2.) There are several different proofs of this fact in the literature. For example, [6, 7, 10] contain proofs using Functional Analysis methods. Another proof follows from a theorem of Malgrange [8]. There is also an unpublished proof by Lojasiewicz [10] based on the theory of semianalytic sets.

The focus of this note is to find generalizations of this result to the case of systems of real analytic equations. More precisely, given a system of real analytic equations $p_j(x,y) = 0, 1 \leq j \leq l$, $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_m)$, under what conditions can one conclude that a $\mathcal{C}^\infty$ or formal solution $y = f(x)$ is necessarily real analytic? Such questions arise naturally in the study of reflection principles in several complex variables, see e.g. [2, 3, 4, 9]. The earliest results on analyticity of formal and $\mathcal{C}^\infty$ solutions of real analytic equations were obtained by Artin [1] and Malgrange [8] respectively.

Since the question of analyticity is local, we will use the language of germs. Unless specified otherwise, functions, maps, spaces etc. are to be understood as germs at 0 of functions, maps, spaces etc.

Let $\mathcal{I}$ be a nonzero ideal in the ring (of germs at 0) of real analytic real valued functions in the variables (x,y), $x = (x_1, x_2, \dots, x_n)$, $y=(y_1, y_2, \dots, y_m)$. Define $\mathrm{Jac}_m(\mathcal{I})$ to be the analytic ideal generated by $\mathcal{I}$ and all the $m \times m$ minors of the Jacobians, $\mathrm{Jac}(Q_1, Q_2, \dots, Q_m)$, where $Q_1, Q_2, \dots, Q_m \in \mathcal{I}$. Set $\mathcal{I}_0 = \mathcal{I}$, $\mathcal{I}_{k+1} = \mathrm{Jac}_m(\mathcal{I}_k)$ for all $k \geq 0$, and $\mathcal{I}_\infty = \cup_{k=0}^\infty \mathcal{I}_k$. A $\mathcal{C}^\infty$ (*resp.* formal power series) map $y = f(x)$, $f(0) = 0$, is called a $\mathcal{C}^\infty$ (*resp.* formal) solution of the ideal $\mathcal{I}$ if

$$Q(x, f(x)) \equiv 0, \ \forall Q \in \mathcal{I}. \tag{1}$$

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For a $\mathcal{C}^\infty$ map $y = f(x)$, let $\mathcal{J}_f$ denote the (proper) ideal in the ring of formal power series in (x, y) generated by $y_j - F_j(x)$, $1 \le j \le m$, where F_j denotes the Taylor expansion at 0 of f_j.

Theorem 1. *Any $\mathcal{C}^\infty$ or a formal solution $y = f(x)$ of a nonzero ideal $\mathcal{I} \subset \mathcal{O}$ satisfying $\mathcal{I}_\infty \not\subseteq \mathcal{J}_f$ is necessarily real analytic.*

Proof. Let $y = f(x)$ be a solution of $\mathcal{I}$, and let k be an integer such that $\mathcal{I}_k \subseteq \mathcal{J}_f$ but $\mathcal{I}_{k+1} \not\subseteq \mathcal{J}_f$. By replacing $\mathcal{I}$ by $\mathcal{I}_k$, if necessary, we may assume that $\operatorname{Jac}_m(\mathcal{I}) \not\subseteq \mathcal{J}_f$. That means there exist Q_1,Q_2,…,Q_m in $\mathcal{I}$ such that

$$\det \frac{\partial Q}{\partial(x', y')}(x, f(x)) \not\equiv 0, \tag{2}$$

where x' and y' are, respectively, blocks of p of the x-variables and $m-p$ of the y-variables . By reordering the y-variables we may assume $y' = (y_{p+1}, y_{p+2}, \dots, y_m)$. Similarly, we may assume that $x' = (x_1, x_2, \dots, x_p)$. Since $Q_i(x, f(x)) \equiv 0$, we have

$$\frac{\partial Q_i}{\partial x_j}(x, f(x)) + \sum_{k=1}^{m} \frac{\partial Q_i}{\partial y_k}(x, f(x)) \cdot \frac{\partial f_k}{\partial x_j} \equiv 0,\ \forall i, j, 1 \le i \le m, 1 \le j \le p. \tag{3}$$

Define an $m \times m$ matrix

$$T = \left[\begin{array}{cc} \frac{\partial f}{\partial x'} & A \end{array}\right],\ A = \left[\begin{array}{c} O_{p\times m-p} \\ -I_{m-p\times m-p} \end{array}\right],$$

where O and I denote the zero and identity matrix, respectively.

By (3), we have the following matrix equality:

$$\frac{\partial Q}{\partial(x', y')}(x, f(x)) + \frac{\partial Q}{\partial y}(x, f(x)) \cdot T \equiv 0.$$

Hence, by (2), we have

$$\det \frac{\partial Q}{\partial y}(x, f(x)) \not\equiv 0. \tag{4}$$

Now we can write

$$Q(x, y) - Q(x, z) = U(x, y, z) \cdot (y - z), \tag{5}$$

where $U(x, y, z)$ is an $m \times m$ matrix with analytic entries satisfying $U(x, y, y) = \frac{\partial Q}{\partial y}(x, y)$. (For example take $U(x, y, z) = \int_0^1 \frac{\partial Q}{\partial y}(x, ty + (1-t)z)dt$.) By (4), we get $\det U(x, F(x), F(x)) \not\equiv 0$. This implies that there exists an integer k such that for any power series $h(x)$, we have

$$F(x) - h(x) \equiv 0 \, mod \, (x)^k \ \Rightarrow\ \det U(x, F(x), h(x)) \not\equiv 0.$$

(Here (x) denotes the ideal generated by $x_1, x_2, \dots, x_n$.) Since $y = F(x)$ is a formal solution of $Q(x, y) = 0$, the Artin Approximation Theorem [1, 11], when applied to the above integer k, yields an analytic solution $y = h(x)$ of $Q(x, y) = 0$ such that $F(x) - h(x) \equiv 0 \, mod \, (x)^k$. Let

$$d(x) := \det U(x, f(x), h(x)),$$

and let $D(x)$ denote its Taylor expansion at 0. Then $D(x) \not\equiv 0$, and hence $d(x)$ has nonzero Taylor expansion at all points near 0. By (5), $0 = U(x, f(x), h(x)) \cdot (f - h)$, and hence $0 = d(x) \cdot (f(x) - h(x))$. But the Taylor expansions of $d(x)$ at points near

0 are all nonzero and the formal powers series expansions ring has no zero divisors. Therefore $f(x) = h(x)$ (in the sense of germs at 0), as required.

Corollary 1. *If* $1 \in \mathcal{I}_\infty$*, then any* $\mathcal{C}^\infty$ *or formal solution of* $\mathcal{I}$ *is real analytic.*

Proof. The ideal $\mathcal{J}_f$ is proper for any solution f since, by definition of a solution, $f(0) = 0$. In particular, $1 \notin \mathcal{J}_f$ and the corollary follows. □

The condition in Corollary 1 is independent of the particular solution. In that sense Corollary 1 is a correct generalization of the result concerning a single equation.

Corollary 2 (cf. [8, 6, 7, 10])**.** *Let* $f(x)$*,* $x = (x_1, x_2, \ldots x_n)$*,* $f(0) = 0$*, be a* $\mathcal{C}^\infty$ *function or a formal power series. If there is a nonzero real analytic function* $\Phi(x,t)$ *such that* $\Phi(x, f(x)) \equiv 0$ *then* $f \in \mathcal{C}^\omega$*.*

Proof. If $\mathcal{I}$ denotes the ideal generated by Φ, then $\mathcal{I}_\infty$ is generated by all the derivatives of Φ. Since Φ is nonzero and real analytic, some derivative of Φ is nonzero at $(x,t) = (0,0)$. Hence $1 \in \mathcal{I}_\infty$, and Corollary 1 applies.

The condition in Theorem 2 below, although solution specific, is natural and is easy to verify for a given solution.

Theorem 2. *Let* $h : \mathbb{R}^n \longrightarrow \mathbb{R}^m$*,* $h(0) = 0$*, be a* $\mathcal{C}^\infty$ *map, and let* $q : \mathbb{R}^m \longrightarrow \mathbb{R}^m$*,* $q(0) = 0$*, be a real analytic map. Let* $\triangle = \det \left(\frac{\partial q_j}{\partial \xi_k}\right)_{m\times m}$*. If the following two conditions are satisfied*

(i) $q \circ h \in \mathcal{C}^\omega(\mathbf{R}^n)$,
(ii) $\triangle \circ h \not\equiv 0$,
then $h \in \mathcal{C}^\omega(\mathbf{R}^n)$*.*

Proof. Observe that if $\Delta(h(p)) \neq 0$ for some p, then by applying the Implicit Function Theorem to $q(\xi) = q(h(x))$, we conclude that $\xi = h(x)$ is the unique real analytic solution near p. Hence h is real analytic in the set $\{x : \Delta(h(x)) \neq 0\}$. □

Case I. $h \sim 0$. (The symbol $\sim$ means that the Taylor expansion at the origin is zero.)

We will show that $h \equiv 0$ in this case.

By (i), we have $q(h(x)) \equiv 0$. If p_0 is any fixed point such that $\triangle(h(p_0)) \neq 0$, then, by arguing as in the above observation, we conclude that $h(x) = h(p_0)$ for all x near p_0.

Hence, the set

$$U = \{x \ : \ h(y) = h(p_0) \ \forall \ y \text{ near } x\}$$

is nonempty and is clearly open. We claim that U is also closed. Indeed, by the continuity of h, we have for any $\tilde{x}$ in the closure of U, $h(\tilde{x}) = h(p_0)$. Also, we have $\triangle(h(\tilde{x})) = \triangle(h(p_0)) \neq 0$, and hence $\tilde{x} \in U$. Hence $U = \mathbb{R}^n$. Since $h(0) = 0$, we have $h(x) = 0$ for all x.

Case II. $h \not\sim 0$.

We claim that it is enough to prove that $\triangle \circ h \not\sim 0$. Indeed, if $\triangle(H(x)) \not\equiv 0$, where H is the Taylor expansion of h at 0, then Theorem 1 implies that $h(x)$ is real analytic.

By an appropriate rotation around 0, we may assume that $\delta(t) = \triangle \circ h_0(t) \not\equiv 0$ where $h_0(t) = h(t, 0, \ldots, 0)$. It is enough to prove that $\delta \not\sim 0$. Since $\delta(t) \not\equiv 0$ near

0, by making the transformation $t \to -t$, if necessary, we may assume that $\delta(t) \not\equiv 0$ on $[0, \varepsilon]$ for all $\varepsilon > 0$.

Lemma. *There exists an $\varepsilon > 0$ such that $\delta(t) \neq 0$ for all $t \in (0, \varepsilon]$.*

Proof. Since the function $q \circ h$ is real analytic , the set

$$Z = \{(\xi, t) \in \mathbb{R}^m \times (0, \varepsilon) : q(\xi) = q \circ h(t)\}$$

is semianalytic. Let Z' denote the set all smooth points $\zeta \in Z$ with $\dim_\zeta(Z) = 1$. (All results about semianalytic sets used here can be found in [5].)

Let $J \subset (0, \varepsilon)$ be any open interval such that $\delta(t) \neq 0$ for all $t \in J$. Observe that the Jacobian determinant of the map

$$\mathbb{R}^m \times J \ni (\xi, t) \to (q(\xi), t)$$

at a point $(h_0(t), t)$ is equal to $\delta(t)$. Hence, the above map is a real analytic diffeomorphism of a neighbourhood of the graph of h over J onto a neighbourhood of the graph of $q \circ h$ over J. We conclude that $(h_0(t), t) \in Z'$ for all t where $\delta(t) \neq 0$.

Note that $\delta(t)$ real analytic on the set $\{t : \delta(t) \neq 0\}$ because $h_0(t)$ is real analytic there. Since Z' is a 1-dimensional smooth semianalytic set, $Z'' = Z' \cap \{(\xi, t) : \Delta(\xi) \neq 0\}$ is also a semianalytic set. Observe that a connected component of the graph of h over the set $\{t \in (0, \varepsilon) : \delta(t) \neq 0\}$ is also a connected component of Z''. Since the family of connected components of a semianalytic set is locally finite, $\delta(t)$ can have only a finite number of zeroes in $(0, \varepsilon)$. By choosing $\varepsilon > 0$ sufficiently small, the lemma follows. □

Let $\varepsilon > 0$ such that $\delta(t) \neq 0$ for all $t \in (0, \varepsilon]$.

The graph of h over $(0, \varepsilon]$, being a connected component of Z'', is a semianalytic set. Hence, the closure

$$G = \{(\xi, t) \in \mathbb{R}^m \times [0, \varepsilon] \ : \xi = h_0(t)\},$$

is also semianalytic.

We claim that it is enough to restrict to the case when h satisfies

$$|h_0(t)| \geq |t| \ \forall t \text{ near } 0. \tag{6}$$

Indeed, if we put $h^\star(x) = (h(x), x)$ and $q^\star(\zeta) = (q(\xi), \tau)$, $\zeta = (\xi, \tau), \tau \in \mathbb{R}$, then $q^\star \circ h^\star \in \mathcal{C}^\omega$. Clearly $|h^\star(t)| \geq |t|$ near 0, where $h_0^\star(t) = (h_0(t), t)$, and moreover, $\delta(t) = \det\left(\frac{\partial q^\star}{\partial \zeta}\right)(h_0^\star(t))$ for all t.

By the lemma, the intersection of the one-dimensional semianalytic set G with the analytic set $V = \{\xi : \Delta(\xi) = 0\}$ is $0 = h_0(0)$. By a basic inequality (due to Lojasiewicz) in the theory of semianalytic sets, there are a constant $C_1 > 0$ and an integer $l_1 > 0$, see e.g. [5], such that

$$dist(h_0(t), V) \geq C_1 \left(dist(h_0(t), G \cap V)\right)^{l_1} = C_1 \left(dist(h_0(t), 0)\right)^{l_1}. \tag{7}$$

Also by Lojasiewicz's Inequality, there are a constant $C_2 > 0$ and an integer $l_2 > 0$ such that

$$|\Delta(\xi)| \geq C_2 \cdot dist(\xi, V)^{l_2}. \tag{8}$$

By combining (6), (7), and (8), we conclude that there are a $t_0 > 0$, $C > 0$, and an integer $l > 0$ such that

$$|\delta(t)| \geq C|t|^l, \ \forall t \in [0, t_0].$$

Since δ is $\mathcal{C}^\infty$, it is easy to see that above inequality implies that the k-th derivative $\delta^{(k)}(0) \neq 0$ for some k, $0 \leq k \leq l$. $\square$

Remark 1. Above results are no longer true if the hypothesis $f \in \mathcal{C}^\infty(\mathbf{R}^n)$ is replaced by $f \in \mathcal{C}^l(\mathbf{R}^n)$, $l < \infty$. For example, take $\phi(y) = y^2$ and $f(x) = x^{\frac{5}{2}}$.

Remark 2. [5] contains the following interesting and related result:

Let $h = (h_1, h_2, \dots, h_m)$, where $h_j(x)$, $1 \leq j \leq n$, are convergent power series in n variables , and $Q(y)$ a formal power series in m variables. If the generic rank of h is m, then

$$Q(h) \text{ convergent } \Rightarrow Q \text{ convergent } .$$

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