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OPEN COVERS AND THE SQUARE BRACKET PARTITION RELATION

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Abstract. An open cover $\mathcal{U}$ of an infinite separable metric space X is an ω-cover of X if $X \notin \mathcal{U}$ and for every finite subset F of X there is a $U \in \mathcal{U}$ such that $F \subseteq U$. Let Ω be the collection of ω–covers of X. We show that the partition relation $\Omega \to [\Omega]^2_2$ holds if, and only if, the partition relation $\Omega \to [\Omega]^2_3$ holds.

For a set S and for a positive integer n the symbol $[S]^n$ denotes the collection of n–element subsets of S. Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of objects and let ℓ, k and n be positive integers with $\ell < k$. Then the symbol

$$\mathcal{A} \to [\mathcal{B}]^n_{k/\leq \ell}$$

denotes the statement:

for every element A of $\mathcal{A}$ and for every function

$$f : [A]^n \to \{0, 1, \dots, k-1\}$$

there are a subset B of A and a $J \in [\{0, 1, \dots, k-1\}]^{\leq \ell}$ such that B is an element of $\mathcal{B}$, and $f(X) \in J$ whenever X is an n–element subset of B.

Thus the symbol denotes a binary relation between elements of $\mathcal{A}$ and of $\mathcal{B}$, and is known as "the square bracket partition relation". The negation of this statement is denoted $\mathcal{A} \not\to [\mathcal{B}]^n_{k/\leq \ell}$. In the case when ℓ is $k-1$, we write $\mathcal{A} \to [\mathcal{B}]^n_k$ to denote $\mathcal{A} \to [\mathcal{B}]^n_{k/\leq \ell}$.

Let κ and λ be cardinal numbers. In [5], where these symbols were introduced, $\mathcal{A}$ happened to be the κ–element subsets of κ, while $\mathcal{B}$ happened to be the λ–element subsets of κ. In this particular situation it is more customary to write $\kappa \to [\lambda]^n_{k/\leq \ell}$. A special case of Ramsey's famous theorem ([7], Theorem A) can be stated in this notation as

$$\aleph_0 \to [\aleph_0]^2_2.$$

These square bracket partition relations have been used in several other contexts too: Let $\mathcal{A}$ and $\mathcal{B}$ both be the collection of those subsets of the rational numbers which are order isomorphic to the set of rational numbers. An unpublished theorem

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of Galvin states $\mathcal{A} \to [\mathcal{B}]^2_3$ – or as is more customary, $\eta \to [\eta]^2_3$. It is easy to show that $\eta \not\to [\eta]^2_2$ (for $0 < \frac{p}{q} < \frac{r}{s}$ both fractions in reduced form, define $\{\frac{p}{q}, \frac{r}{s}\}$ to be of color 0 if $2^p \cdot 3^q < 2^r \cdot 3^s$, and to be of color 1 otherwise).

The square bracket partition relation has been extensively studied in connection with ultrafilters. Though these have been studied also for ultrafilters on uncountable sets, I shall restrict my comments here to ultrafilters on the set of positive integers. A nonprincipal ultrafilter $\mathcal{U}$ is said to be a *Ramsey ultrafilter* if it satisfies $\mathcal{U} \to [\mathcal{U}]^2_2$. These are discussed for example in [3]. In [2] Blass introduces the notion of a *weakly Ramsey ultrafilter*: a nonprincipal ultrafilter $\mathcal{U}$ on the set of positive integers is said to be weakly Ramsey if it satisfies the partition relation $\mathcal{U} \to [\mathcal{U}]^2_3$. He then showed that the Continuum Hypothesis implies that there is a weakly Ramsey ultrafilter which is not a Ramsey ultrafilter. In other words, the implication

$$\mathcal{U} \to [\mathcal{U}]^2_2 \Rightarrow \mathcal{U} \to [\mathcal{U}]^2_3 \tag{1}$$

is not provably reversible. Along these lines D. Devlin (using for example the Continuum Hypothesis) gave examples in [4] of nonprincipal ultrafilters $\mathcal{U}$ for which the relation $\mathcal{U} \to [\mathcal{U}]^2_6$ holds, while $\mathcal{U} \to [\mathcal{U}]^2_5$ fails.

Analogues of this situation also occur for cardinal numbers. It is known that when κ is a singular cardinal number, then for all integers $n \geq 2$ $\kappa \not\to [\kappa]^n_{2^n-1}$ (see [11], Theorem 2.7.8), while when κ has countable cofinality, it can happen that $\kappa \to [\kappa]^n_{2^{n-1}+1}$ (see [11], Theorem 2.7.7). The moral of the story seems to be that $\mathcal{A} \to [\mathcal{B}]^2_{k+1}$ does not trivially imply $\mathcal{A} \to [\mathcal{B}]^2_k$.

In [9] partition relations for families of open covers of topological spaces were introduced, and in [6], [9] and [10] it was shown that these characterize some of the classical small sets of real numbers introduced by Rothberger, Menger and Hurewicz. It was also shown that there is a strong analogy between the combinatorial properties of certain collections of open covers for these sorts of spaces, and combinatorial properties of classical sorts of ultrafilters on the set of positive integers.

For example: Let X be an infinite separable metric space. Then X is said to have the *Rothberger property* if there is for every sequence $(\mathcal{U}_n : n < \omega)$ of open covers of X a sequence $(U_n : n < \omega)$ such that for each n $U_n \in \mathcal{U}_n$, and $\{U_n : n < \omega\}$ is an open cover of X. This property was introduced by Rothberger in [8]. Also, call an open cover $\mathcal{U}$ of X an ω–cover if there is for each finite subset F of X an element U of $\mathcal{U}$ such that $F \subseteq U$. Let Ω denote the collection of all ω–covers of X. Results of [6] and [9] show that the partition relation $\Omega \to [\Omega]^2_2$ holds if, and only if, all finite powers of X have Rothberger's property.

In light of what occurs in analogous situations, it is natural to now ask if one could have spaces for which $\Omega \to [\Omega]^2_3$ holds while $\Omega \to [\Omega]^2_2$ fails. The purpose of this note is to show that the answer is "No". The result gives, via the methods of [1], a few more characterizations of those sets whose finite powers have the Rothberger property. The following fact, part of the folklore of Ramsey theory, is used in our proof:

Theorem 1. *Let X be an infinite separable metric space. If $\Omega \to [\Omega]^2_3$, then for every positive integer k, $\Omega \to [\Omega]^2_{k/\leq 2}$.*

Proof. We induct on k. The theorem is true for $k \leq 3$. Assume that $k \geq 3$ and that the implication has been verified up to k. Let $\mathcal{U}$ be an ω–cover of X and let

$f:[\mathcal{U}]^2 \rightarrow \{0,1,\dots,k\}$ be given. Define $g:[\mathcal{U}]^2 \rightarrow \{0,1,2\}$ so that

$$g(\{U,V\}) = \begin{cases} 0 & \text{if } f(\{U,V\}) = 0, \\ 1 & \text{if } f(\{U,V\}) = 1, \\ 2 & \text{if } f(\{U,V\}) > 1. \end{cases}$$

Apply $\Omega \rightarrow [\Omega]^2_3$ to find an ω-cover $\mathcal{V} \subset \mathcal{U}$ on which g is ≤ 2–valued. Then on $\mathcal{V}$ f is $\leq k$–valued, and we may apply the induction hypothesis to find an ω–cover $\mathcal{W} \subset \mathcal{V}$ such that on it f is ≤ 2–valued. □

By the results of [6] and [9] an infinite separable metric space's family of ω–covers satisfies the partition relation $\Omega \rightarrow [\Omega]^2_2$ if, and only if, it has the following Rothberger–like covering property:

> For every sequence $(\mathcal{U}_n : n < \omega)$ of ω–covers of X there is a sequence $(U_n : n < \omega)$ such that for each n $U_n \in \mathcal{U}_n$, and such that $\{U_n : n < \omega\}$ is an ω–cover of X.

For further use below, it is worth noting at this point that if $\mathcal{U}$ is an ω–cover of X, then no finite subset of $\mathcal{U}$ is an ω–cover of X (even though it might be a cover). If $\mathcal{U}$ is an ω-cover of X which refines the open cover $\mathcal{V}$ of X, and if X is not an element of $\mathcal{V}$, then $\mathcal{V}$ is also an ω–cover of X.

Theorem 2. *If X is an infinite separable metric space, then the following statements are equivalent:*

1. $\Omega \rightarrow [\Omega]^2_2$.
2. $\Omega \rightarrow [\Omega]^2_3$.

Proof. We must show that 2 implies 1. I use the equivalent form of the partition relation, formulated above in terms of sequences of ω-coverings. Thus, let $(\mathcal{U}_n : n < \omega)$ be a sequence of ω-covers of X. Since X is a separable metric space, each ω–cover has a countable refinement which is still an ω–cover. Thus we may assume that each of the $\mathcal{U}_n$'s is countable. For each n, enumerate $\mathcal{U}_n$ bijectively as $(U^n_m : m < \omega)$. Define a new ω–cover $\mathcal{V}$ by

$$\{U^0_k \cap U^k_n \cap U^n_\ell : k,n,\ell < \omega\} \setminus \{\emptyset\}.$$

For each element V of $\mathcal{V}$ choose a representation $V = U^0_k \cap U^k_n \cap U^n_\ell$. The term U^k_n will be called *the middle term* of the representation, while the term U^n_ℓ will be called *the end–term* of the representation. Define a partition $f:[\mathcal{V}]^2 \rightarrow \{0,1,2,3,4\}$ as follows: (in the displayed formula we are assuming that the two arguments of f are listed so that (k_1,n_1) lexicographically precedes (k_2,n_2))

$$f(\{U^0_{k_1} \cap U^{k_1}_{n_1} \cap U^{n_1}_{\ell_1}\},\{U^0_{k_2} \cap U^{k_2}_{n_2} \cap U^{n_2}_{\ell_2}\}) = \begin{cases} 0 & \text{if } k_1 = k_2 \text{ and } n_1 = n_2, \\ 1 & \text{if } k_1 = k_2 \text{ and } n_1 < n_2, \\ 2 & \text{if } k_1 < k_2 \text{ and } n_1 < n_2, \\ 3 & \text{if } k_1 < k_2 \text{ and } n_1 > n_2, \\ 4 & \text{if } k_1 < k_2 \text{ and } n_1 = n_2. \end{cases}$$

By Theorem 1 and by 2, $\Omega \rightarrow [\Omega]^2_{5/\leq 2}$ holds. Choose an ω-cover $\mathcal{W} \subset \mathcal{V}$ on which f is two-valued. List $\mathcal{W}$ as

$$(U^0_{k_1} \cap U^{k_1}_{n_1} \cap U^{n_1}_{\ell_1}, U^0_{k_2} \cap U^{k_2}_{n_2} \cap U^{n_2}_{\ell_2}, \dots)$$

according to the lexicographic order of the triples (k_i,n_i,ℓ_i) which occur in the chosen representations of elements of $\mathcal{W}$. There are four main cases to be considered.

Case 1. $\{0,1\}\cap f[[\mathcal{W}]^2]=\emptyset$. In this case we have $k_1<k_2<\cdots<k_n<\dots$, and so if we set $V_{k_i}=U^{k_i}_{n_i}$, and for $m\notin\{k_i:i=1,2,3,\dots\}$ we choose $V_m\in\mathcal{U}_m$ arbitrarily, then the sequence $(V_m:m=1,2,3,\dots)$ constitutes an ω-cover of X and for each m we have $V_m\in\mathcal{U}_m$.

Case 2. $\{0,1\}\cap f[[\mathcal{W}]^2]=\{1\}$.

Subcase 2.1. $f[[\mathcal{W}]^2]\subseteq\{1,2\}$. In this case we see that $n_i\neq n_j$ whenever $i\neq j$. But then $\mathcal{W}$ refines the sequence $(U^{n_i}_{\ell_i}:i=1,2,3,\dots)$, whence the latter constitutes an ω–cover of X. This sequence can then be augmented to one of the form $(U_n:n<\omega)$ where for each n, $U_n\in\mathcal{U}_n$.

Subcase 2.2. $f[[\mathcal{W}]^2]\subseteq\{1,3\}$.
This case doesn't occur. To see this, first observe that 3 cannot be attained with only finitely many different k's, since then $\mathcal{W}$ would be a refinement of a finite subset of $\mathcal{U}_0$, and thus not an ω-cover. But for Subcase 2.2, 3 also cannot be attained with infinitely many k's. To see this, list the subscripts of the U^0_k's occurring in the representations of elements of $\mathcal{W}$ monotonically, say

$$k_1\leq k_2\leq\cdots\leq k_j\leq\dots.$$

If 3 occurs with infinitely many k's, then it happens infinitely often that $k_j<k_{j+1}$. Define $i_0=1$ and $i_{n+1}>i_n$ to be minimal such that $k_{i_{n+1}}>k_{i_n}$. Look at the subset $\{U^0_{k_{i_j}}\cap U^{k_{i_j}}_{n_{i_j}}\cap U^{n_{i_j}}_{\ell_{i_j}}:j=1,2,3,\dots\}$ of $\mathcal{W}$. It is homogeneous of color 3, and thus we find

$$n_{i_1}>n_{i_2}>\cdots>n_{i_j}>\dots,$$

an infinite descending sequence of natural numbers, a contradiction.

Subcase 2.3. $f[[\mathcal{W}]^2]\subseteq\{1,4\}$. If the value 4 is taken with only finitely many k's, then $\mathcal{W}$ would be a refinement of a finite subset of $\mathcal{U}_0$, and thus not an ω-cover of X. If the value 1 is taken with only finitely many k's, then all but finitely many of the middle terms $U^{k_i}_{n_i}$ could be assigned to distinct ones of the covers $\mathcal{U}_n$, and would constitute an ω–cover, in which case we would be done. Thus we must treat the case where each of 1 and 4 is attained with infinitely many k's. Then

$$(k_1,n_1),(k_2,n_2),\dots,(k_r,n_r),\dots$$

forms (in the lexicographic order) a strictly increasing sequence such that for each i, either $k_i<k_{i+1}$ and $n_i=n_{i+1}$ (in which case $f(\{U^0_{k_i}\cap U^{k_i}_{n_i}\cap U^{n_i}_{\ell_i},U^0_{k_{i+1}}\cap U^{k_{i+1}}_{n_{i+1}}\cap U^{n_{i+1}}_{\ell_{i+1}}\})=4$), or else $k_i=k_{i+1}$ and $n_i<n_{i+1}$ (in which case $f(\{U^0_{k_i}\cap U^{k_i}_{n_i}\cap U^{n_i}_{\ell_i},U^0_{k_{i+1}}\cap U^{k_{i+1}}_{n_{i+1}}\cap U^{n_{i+1}}_{\ell_{i+1}}\})=1$). Consider the two sequences

$$(U^{k_i}_{n_i}:i=1,2,3,\dots\text{ and }k_{i-1}<k_i\text{ or }k_i<k_{i+1})$$

which consists of certain middle terms of the three-set intersections composing the elements of $\mathcal{W}$, and

$$(U^{n_i}_{\ell_i}:i=1,2,3,\dots\text{ and }n_{i-1}<n_i\text{ or }n_i<n_{i+1})$$

which consists of certain end terms of the three-set intersections composing the elements of $\mathcal{W}$. Since $\mathcal{W}$ refines the totality of sets belonging to these two sequences, these two sequences constitute an ω–cover of X. For each m the set $\{i:k_i=m\text{ or }n_i=m\}$ has at most four elements.

Thus this ω–cover can be partitioned into four–element sets each of which could be assigned to distinct terms of the original sequence $(\mathcal{U}_n : n < \omega)$. Being an ω–cover, we can find a new ω–cover by selecting one term per each of these four–element sets (*i.e.*, $\Omega \to (\Omega)^1_4$ holds). In this way we find a selector for the original sequence of ω–covers in such a way that the selector is also an ω–cover.

Case 3. $\{0,1\} \cap f[[\mathcal{W}]^2] = \{0\}$.

Subcase 3.1. $f[[\mathcal{W}]^2] \subseteq \{0,2\}$.
If the value 2 occurs at only a finite number of distinct k's, then $\mathcal{W}$ is a refinement of a finite subset of $\mathcal{U}_0$, and thus not an ω-cover of X. So, the value 2 is achieved at infinitely many k's. Each time it is achieved and only then, both k_i and n_i increase in value. Let $i_1 < i_2 < \dots$ be such that

- $i_1 = 1$,
- for each j, if $i_j \leq t < i_{j+1}$, then $k_{i_j} = k_t$ and $n_{i_j} = n_t$,
- for each j, $k_{i_j} < k_{i_{j+1}}$, and
- $\{(k_{i_j}, n_{i_j}) : j = 1,2,3,\dots\} = \{(k_j, n_j) : j = 1,2,3,\dots\}$.

For each j put $V_{k_{i_j}} = U^{k_{i_j}}_{n_{i_j}}$ and for n not in $\{k_{i_j} : j = 1,2,3,\dots\}$, choose V_n from $\mathcal{U}_n$ arbitrarily. Then for each n $V_n \in \mathcal{U}_n$, and $\mathcal{W}$ is a refinement of $\{V_n : n = 1,2,3,\dots\}$; thus the latter is an ω–cover of X.

Subcase 3.2. $f[[\mathcal{W}]^2] \subseteq \{0,3\}$. The value 3 can be attained at only a finite number of distinct k's, lest we have an infinite descending sequence of natural numbers. But then $\mathcal{W}$ is a refinement of a finite subset of $\mathcal{U}_0$, and so not an ω–cover of X. It follows that this case doesn't occur.

Subcase 3.3. $f[[\mathcal{W}]^2] \subseteq \{0,4\}$. The value 4 must be attained at an infinite number of distinct k's, else $\mathcal{W}$ would be a refinement of a finite subset of $\mathcal{U}_0$, hence not an ω-cover of X. But then an argument as in Subcase 3.1 shows that there is a sequence $(U_n : n < \omega)$ such that for each n $U_n \in \mathcal{U}_n$, and $\{U_n : n < \omega\}$ is an ω–cover of X.

Case 4. $f[[\mathcal{W}]^2] \subseteq \{0,1\}$.

Then there is a fixed k such that each element of $\mathcal{W}$ is a subset of U^0_k. Since $X \neq U^0_k$ it follows that $\mathcal{W}$ doesn't even cover X. Consequently, this case doesn't occur. □

Using Theorem 2 one now obtains characterizations analogous to those in Theorem 2.4 of [1] for those sets of reals all of whose finite powers have Rothberger's property.

References

[1] J.E. Baumgartner and A.D. Taylor, *Partition theorems and ultrafilters*, **Transactions of the American Mathematical Society** 241 (1978), 283 – 309. MR **58:**10458

[2] A.R. Blass, *Ultrafilter mappings and their Dedekind cuts*, **Transactions of the American Mathematical Society** 188 (1974), 327 – 340. MR **50:**4310

[3] D. Booth, *Ultrafilters on a countable set*, **Annals of Mathematical Logic** 2 (1970/71), 1 – 24. MR **43:**3104

[4] D. Devlin, *Some partition theorems and ultrafilters on ω*, Ph. D. thesis, Dartmouth College (1979)

[5] P. Erdös, A. Hajnal and R. Rado, *Partition relations for cardinal numbers*, **Acta Math. Acad. Sci. Hungar.** 16 (1965), 93–196. MR **34:**2475
[6] W. Just, A.W. Miller, M. Scheepers and P.J. Szeptycki, *Combinatorics of Open Covers II*, **Topology and its Applications** 73 (1996), 241–266.
[7] F.P. Ramsey, *On a problem of formal logic*, **Proceedings of the London Mathematical Society** 30 (1930), 264 – 286.
[8] F. Rothberger, *Eine Verschärfung der Eigenschaft* C, **Fundamenta Mathematicae** 30 (1938), 50 – 55.
[9] M. Sheepers, *Combinatorics of Open Covers I: Ramsey theory*, **Topology and its Applications** 69 (1996), 31–62. CMP 96:09
[10] M. Scheepers, *Rothberger's property and partition relations*, **The Journal of Symbolic Logic**, to appear.
[11] N.H. Williams, *Combinatorial Set Theory*, **North-Holland** (1980).

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