PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 9, September 1997, Pages 2743–2751
S 0002-9939(97)03904-X

STABLE ORDERS OF STUNTED LENS SPACES MOD 2^v

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(Communicated by Thomas G. Goodwillie)

ABSTRACT. Let L_{2n-1}^{2n+2m} be the stunted lens space mod 2^v and $|L_{2n-1}^{2n+2m}|$ its stable order. If $v = 1$, then $|L_{2n-1}^{2n+2m}|$ was determined by H. Toda (1963). In this paper, we determine the number $|L_{2n-1}^{2n+2m}|$ for $v \geq 2$.

1. INTRODUCTION

The stable order of a finite torsion spectrum X, denoted by $|X|$, is the order of the stable identity map in the group $[X, X]$ consisting of classes of stable self maps on X. The suspension spectrum of a finite CW-complex is a torsion spectrum if $H^*(X; Z)$ is a torsion group. In this case we define the stable order of X, denoted also by $|X|$, to be the stable order of the suspension spectrum of X.

Let L^{2n+1} be the standard lens space mod 2^v. As indicated in [9, p. 91], there is a CW-structure on L^{2n+1} with just one cell in each dimension $\leq 2n+1$. For the infinite lens space L^∞ mod 2^v, we have $H^*(L^\infty; \mathbf{Z}_2) \approx \mathbf{Z}_2[u, w]/(u^2)$ if $v \geq 2$, where $\deg u = 1$ and $\deg w = 2$, and the Steenrod operations are given by $\mathrm{Sq}^{2i}(w^j) = \binom{j}{i} w^{i+j}$, $\mathrm{Sq}^{2i}(uw^j) = \binom{j}{i} uw^{i+j}$ and $\mathrm{Sq}^{2i+1}(-) = 0$. Let L^{2n} be the $2n$-skeleton of L^{2n+1}, and $L_n^{n+k} = L^{n+k}/L^{n-1}$. If $v = 1$, then L_{2n-1}^{2n+2m} is the stunted real projective space $P_{2n-1}^{2(n+m)}$, and $|P_{2n-1}^{2(n+m)}|$ was determined by H. Toda in [8, p. 300]. In this paper, we prove

Theorem 1.1. *Let L_{2n-1}^{2n+2m} be the stunted lens space mod 2^v with $v \geq 2$ and $m \geq 0$. Then*

$$|L_{2n-1}^{2n+2m}| = \begin{cases} 2^{v+m} & \text{if either } n \text{ is odd or } m \not\equiv 2 \pmod 4, \\ 2^{v+m+1} & \text{if } n \text{ is even and } m \equiv 2 \pmod 4. \end{cases}$$

We compare the above theorem with Toda's result on the case when $v = 1$, which can be read off from [8, p. 300] as follows.

Received by the editors May 25, 1995 and, in revised form, March 13, 1996.
1991 *Mathematics Subject Classification.* Primary 55N15, 55P25, 55T15.
Key words and phrases. K-theory, stunted lens spaces, Adams spectral sequences, vanishing line theorem.

Theorem 1.2. *Let $m \geq 1$. Then*

$$|P_{2n-1}^{2n+2m}| = \begin{cases} 2^{m+1} & \text{if } m \equiv 3 \pmod 4 \text{ or} \\ & \text{if } m \equiv 2 \pmod 4 \text{ and } n \text{ odd or} \\ & \text{if } m \equiv 0 \pmod 4 \text{ and } n \text{ even;} \\ 2^{m+2} & \text{otherwise;} \end{cases}$$

while $|P^2| = 4$ corresponding to $m = 0$.

Remark 1.3. As in [8], L_{2n-1}^{2n+2m} can be replaced by a finite CW-complex X in Theorem 1.1 if there exist isomorphisms

$$f_1 : H^*(L_{2n-1}^{2n+2m}; \mathbf{Z}_{(2)}) \to H^*(X; \mathbf{Z}_{(2)}) \quad \text{and} \qquad f_2 : H^*(L_{2n-1}^{2n+2m}; \mathbf{Z}_2) \to H^*(X; \mathbf{Z}_2)$$

such that $f_2 \operatorname{Sq}^i(-) = \operatorname{Sq}^i f_2(-)$ for $i = 1, 2$. Here $\mathbf{Z}_{(2)}$ is the integers localized at 2.

The paper is organized as follows. In Section 2, K-theory is used to give a lower bound for $|L_{2n-1}^{2(n+m)}|$. In Section 3, we review an algebraic Atiyah-Hirzebruch spectral sequence and some Adams differentials. In Section 4, we consider the exponent of the group π_0 of the lens space $L_{2n-1}^{2(n+m)}$ smashed with its dual by studying the order of the class derived from the identity map on $L_{2n-1}^{2(n+m)}$ (which is equal to $|L_{2n-1}^{2(n+m)}|$). Adams spectral sequence and the vanishing line theorem allow us to give an upper bound for this order and hence for $|L_{2n-1}^{2(n+m)}|$. Then Theorem 1.1 follows easily from the fact that two bounds actually coincide.

I am grateful to Professor Donald M. Davis for his constant encouragement and his critical reading of the manuscript. I should also thank the referee for many valuable comments and suggestions, and Professor Martin Bendersky for many conversations.

2. K^*-cohomology

In this section, we prove Proposition 2.1 and Theorem 2.3, which will be used in Section 4 to detect the estimates for $|L_{2n-1}^{2(n+m)}|$. We assume that $v \geq 2$ throughout the remainder of this paper.

Let η be the complex Hopf bundle over CP^n and $\mu = \eta - 1$, a generator of the ring $\widetilde{K}(CP^n)$. Let $\sigma = \rho^*(\mu)$, where $\rho : L^{2n+1} \to CP^n$ is the standard projection. Denote also by ρ the stable map $L_{2n-1}^{2n+2m+1} \to CP^{n+m}/CP^{n-1} = CP_n^{n+m}$ or $L_{2n-1}^{2n+2m} \to CP_n^{n+m}$ induced by the standard projection $\rho : L^{2n+2m+1} \to CP^{n+m}$ or $L^{2n+2m} \to CP^{n+m}$. For a cohomology theory $E^*(-)$, let $E_2^{p,q}(E^*(X))$ denote the E_2-term of the Atiyah-Hirzebruch spectral sequence converging to $E^*(X)$.

Proposition 2.1. (i) *$\widetilde{K}(L_{2n-1}^{2n+2m+1}) \approx \widetilde{K}(L_{2n-1}^{2n+2m})$. For $1 \leq i \leq [k/2]$, the order of σ^i in $\widetilde{K}(L^k)$ is $2^{v+[k/2]-i}$.*

(ii) *The element $\sigma^{n+t} \in \widetilde{K}(L_{2n-1}^{2n+2m})$ is of order 2^{v+m-t} for $0 \leq t \leq m$.*

Proof. The second half of (i) can be found in [4, p. 85], while the first half follows easily from the fact that both Atiyah-Hirzebruch spectral sequences converging to $K^*(L_{2n-1}^{2n+2m+1})$ and $K^*(L_{2n-1}^{2n+2m})$ collapse and the map

$$E_2^{p,-p}(K^*(L_{2n-1}^{2n+2m+1})) \to E_2^{p,-p}(K^*(L_{2n-1}^{2n+2m}))$$

induced by the inclusion $L_{2n-1}^{2n+2m} \to L_{2n-1}^{2n+2m+1}$ is an isomorphism.

For (ii), the Atiyah-Hirzebruch spectral sequences converging to $K^*(L^{2n+2m})$ and $K^*(L_{2n-1}^{2n+2m})$ also collapse. Thus we can easily see by looking into the E_2 terms that $p^* : K^0(L_{2n-1}^{2n+2m}) \to K^0(L^{2n+2m})$ is injective, where p is the projection $L^{2n+2m} \to L_{2n-1}^{2n+2m}$. Thus (ii) follows from (i). □

Lemma 2.2. *(i)* $\rho^* : \widetilde{K}(CP_n^{n+m}) \to \widetilde{K}(L_{2n-1}^{2n+2m})$ *is surjective.*

(ii) $2^{v+m}x = 0$ *for* $x \in \widetilde{K}(L_{2n-1}^{2n+2m})$.

Proof. For (i), the Atiyah-Hirzebruch spectral sequences converging to $K^*(L_{2n-1}^{2n+2m})$ and $K^*(CP_n^{n+m})$ collapse. Since ρ^* is surjective in $H^*(-;\mathbf{Z})$, the map

$$\rho^* : E_2^{p,-p}(K^*(CP_n^{n+m})) \to E_2^{p,-p}(K^*(L_{2n-1}^{2n+2m}))$$

induced by ρ is surjective. Thus (i) follows.

Consider (ii). By [1, p. 621], the group $\widetilde{K}(CP_n^{n+m})$ is generated freely by $\mu^n, \mu^{n+1}, \dots, \mu^{n+m}$. Thus by (i), the group $\widetilde{K}(L_{2n-1}^{2n+2m})$ is generated by $\sigma^n, \sigma^{n+1}, \dots, \sigma^{n+m}$. Then (ii) follows from Proposition 2.1 (ii). □

For a group G, let $|x|$ denote the order of $x \in G$.

Theorem 2.3. *(i) Let* $x \in \widetilde{K}(L_{2n-1}^{2n+2m})$ *be such that the restriction of* x *on* L_{2n-1}^{2n} *generates* $\widetilde{K}(L_{2n-1}^{2n}) \approx \mathbf{Z}/2^v$. *Then* $|x| = 2^{v+m}$.

(ii) Let $x \in K^{4k}(L_{4k-1}^{4k+8l+4})$ *with* $|x| = 2^{v+4l+2}$. *Then the realification*

$$r : K^{4k}(L_{4k-1}^{4k+8l+4}) \to KO^{4k}(L_{4k-1}^{4k+8l+4})$$

satisfies $|r(x)| = 2^{v+4l+2}$.

(iii) Let $x \in KO^{4k}(L_{4k-1}^{4k+8l+4})$ *be such that the restriction of* x *on* L_{4k-1}^{4k} *generates* $KO^{4k}(L_{4k-1}^{4k}) \approx \mathbf{Z}/2^v$. *Then* $|x| = 2^{v+4l+3}$.

Proof. Consider (i). Suppose $\sigma^n = (kx)$ for an odd integer k when restricted on L_{2n-1}^{2n}. It suffices to prove $|kx| = 2^{v+m}$. Since $kx - \sigma^n = p^*(y)$ for some $y \in \widetilde{K}(L_{2n+1}^{2n+m})$, where p is the projection $L_{2n-1}^{2n+2m} \to L_{2n+1}^{2n+2m}$, by Lemma 2.2, we have $|y| < 2^{v+m}$. Since $|\sigma^n| = 2^{v+m}$, we conclude that $kx = \sigma^n + p^*(y)$ is of order 2^{v+m}.

For (ii), consider the diagram below:

$$\begin{array}{ccccc} K^{4k}(L_{4k-1}^{4k+8l+3}) & \xleftarrow{i^*} & K^{4k}(L_{4k-1}^{4k+8l+4}) & \xleftarrow{p^*} & K^{4k}(S^{4k+8l+4}) \\ \downarrow r & & \downarrow r & & \downarrow r \\ KO^{4k}(L_{4k-1}^{4k+8l+3}) & \xleftarrow{i^*} & KO^{4k}(L_{4k-1}^{4k+8l+4}) & \xleftarrow{p^*} & KO^{4k}(S^{4k+8l+4}) \end{array}$$

Since $2^{v+4l+1}x$ is of order 2 and is null in $K^{4k}(L_{2n-1}^{2n+8l+3})$, there is a generator $y \in K^{4k}(S^{4k+8l+4})$ such that $p^*(2^{v-1}y) = 2^{v+4l+1}x$. Thus

$$r(2^{v+4l+1}x) = p^*(r(2^{v-1}y)).$$

Since $r : K^{4k}(S^{4k+8l+4}) \to KO^{4k}(S^{4k+8l+4})$ is an isomorphism by [7, p.304], we conclude that $p^*(r(2^{v-1}y))$, and hence also $r(2^{v+4l+1}x)$, is of order 2. Therefore $r(x)$ is of order 2^{v+4l+2}.

Consider (iii). First note that each element of $E_2^{p,q}(KO^*(L_{4k-1}^{4k+8l+4}))$ survives to E_∞ if $p+q \equiv 0 \pmod 4$ by [5, p.240, (1)]. Let c be the complexification. It is well known that $rc = 2$. The restriction of $c(x)$ on L^{4k}_{4k-1} generates $K^{4k}(L^{4k}_{4k-1})$ because $c : \pi_{8t}(KO)\to\pi_{8t}(K)$ is an isomorphism, which implies that the map

$$c : E_2^{4k,0}(KO^*(L^{4k+8l+4}_{4k-1}))\to E_2^{4k,0}(K^*(L^{4k+8l+4}_{4k-1}))$$

induced by the complexification c on coefficients is an isomorphism. Thus $|c(x)| = 2^{v+4l+2}$ by (i), and $rc(x) = 2x$ is of order 2^{v+4l+2} by (ii). Therefore x is of order 2^{v+4l+3}. □

3. An algebraic Atiyah-Hirzebruch spectral sequence and some Adams differentials

Let p be a prime and A the mod p Steenrod algebra. Suppose W is an A-module. Let W_i be the submodule of W consisting of classes of degree $\geq i$. Define $W^i = W/W_{i+1}$ and $W^i_i = W_i/W_{i+1}$. Then we have an exact sequence

$$0\longleftarrow W^{i-1}\longleftarrow W^i\longleftarrow W^i_i\longleftarrow 0$$

of A-modules. Then an algebraic Atiyah-Hirzebruch spectral sequence converging to $\mathrm{Ext}_A(W, \mathbf{Z}_p)$ is defined as in [6] with

$$E_1^{s,t} = \bigoplus_{i=0}^{\infty} \mathrm{Ext}_A^{s,t}(W^i_i, \mathbf{Z}_p),$$

where the differential d_r goes from the $\mathrm{Ext}_A^{s,t}(W^j_j, \mathbf{Z}_p)$ to $\mathrm{Ext}_A^{s+1,t}(W^{j-r}_{j-r}, \mathbf{Z}_p)$.

Let I_1 and I_2 be finite sets. Suppose $X^n_n = \vee_{\alpha\in I_1} S^n_\alpha$ and $X^{n+1}_{n+1} = \vee_{\beta\in I_2} S^{n+1}_\beta$, where S^n_α and S^{n+1}_β are spheres. Let

$$X^n_n\longrightarrow X^{n+1}_n\longrightarrow X^{n+1}_{n+1}$$

be a cofibre sequence. Suppose the attaching map for each $(n+1)$-cell e^{n+1}_β is given by a composite $S^n \xrightarrow{2^v} S^n \xrightarrow{i_\beta} X^n_n$ for a map i_β of Adams filtration 0 and a fixed $v \geq 2$. Let $E_2(X)$ be the E_2-term of the stable mod p Adams spectral sequence for $\pi_*(X)$. Then

$$E_2(X^{n+1}_n) = E_2(X^n_n) \oplus E_2(X^{n+1}_{n+1})$$

since $v \geq 2$. Let $g^s_\beta \in E_2^{s,s+1+n}(X^{n+1}_n)$ generate the summands $E_2^{s,s+1+n}(S^{n+1}_\beta)$ corresponding to the cell e^{n+1}_β and let $g^s_\alpha \in E_2^{s,s+n}(X^{n+1}_n)$ generate $E_2^{s,s+n}(S^n_\alpha)$. Define $w_\beta \in E_2^{0,0}(X^{n+1}_n)$ to be the class corresponding to a class in $E_2^{0,0}(X^n_n)$ surviving to the homotopy class of i_β. Also note that $E_r(-)$ is an $E_r(S^0)$-module by [3, Theorem 2.3.3, p. 63].

Lemma 3.1. *With the above notations, $d_r : E_r^{s,s+1+n}(X^{n+1}_n)\to E_r^{s+r,s+r+n}(X^{n+1}_n)$ is null when $r < v$, and $d_v(g^0_\beta) = h_0^v w_\beta$, where h_0 generates $E_2^{1,1}(S^0)$.*

Proof. First we have

$$E_2^{s,s+1+n}(X^{n+1}_n) = E_2^{s,s+1+n}(X^n_n) \oplus E_2^{s,s+1+n}(X^{n+1}_{n+1})$$

and

$$E_2^{s,s+n}(X^{n+1}_n) = E_2^{s,s+n}(X^n_n).$$

Suppose d_r is the first nontrivial differential $E_r^{0,n+1}(X^{n+1}_n)\to E_r^{r,r+n}(X^{n+1}_n)$ with

$r < v$. Then $d_r(g^0_\beta) = h^r_0(\Sigma_j\ \epsilon_j g^0_{\alpha_j})$ for some $\epsilon_j = 1$. This means that the composite

$$S^n \xrightarrow{2^r} S^n \xrightarrow{i} X^n_n \to X^{n+1}_n$$

is null-homotopic for some map i of Adams filtration 0. This is impossible by the image of the boundary in the exact sequence below

$$\to \pi_{n+1}(X^{n+1}_{n+1}) \xrightarrow{\partial} \pi_n(X^n_n) \to \pi_n(X^{n+1}_n)$$

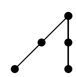
because $r < v$.

For d_v, it suffices to prove the case when $X^{n+1}_{n+1} = S^{n+1}$, a single sphere. Since the composite $S^n \xrightarrow{2^v} S^n \xrightarrow{i_\beta} X^n_n \to X^{n+1}_n$ is null-homotopic, $h^v_0 w_\beta$ must be hit by a differential and it must be a d_v-differential. Therefore $d_v(g^0_\beta) = h^v_0 w_\beta$. □

Lemma 3.2 ([2]). *Let $s \geq [(t-s)/2] + 2$. Then*

$$\mathrm{Ext}^{s,t}_A(\mathbf{Z}_2, \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2 & \text{if } t-s \equiv 3 \pmod 8, \\ 0 & \text{otherwise}; \end{cases}$$

and the chart for $\mathrm{Ext}^{s,t}_A(\mathbf{Z}_2, \mathbf{Z}_2)$ with $8b+1 \leq t-s \leq 8b+3$ and $s \geq 4b+1$ is as follows:



with the two bottom classes corresponding to $s = 4b+1$

4. Proof of Theorem 1.1

In this section $H^*(-)$ always refers to $\mathbf{Z}_2$ coefficients. Recall that $v \geq 2$.

Let X be a finite CW-complex and DX its S-dual. It is well known that $DL^{n+k}_n = \Sigma L^{-n-1}_{-n-k-1}$. Consider the map $\omega' : S^0 \to X \wedge DX$ obtained by S-duality from the identity map $\iota : X \to X$. Let $N = L^{2(n+m)}_{2n-1} \wedge L^{-2n}_{-2(n+m)-1}$. Then we can view N as a CW-complex with one bottom cell (resp. one top cell) of dimension $-2m-2$ (resp. $2m$). Let N^i be the i-skeleton. Define $N^i_j = N^i/N^{j-1}$. There is a map $\omega'' : S^{-1} \to N$ induced by ω'. The composite $S^{-1} \xrightarrow{\omega''} N \to N/N^0$ is null-homotopic, hence ω'' factors as $S^{-1} \xrightarrow{\omega} N^0 \to N$. Since $|\omega'| = |X|$ and $|\omega| \geq |\omega'|$, we will focus on $\omega : S^{-1} \to N^0$.

Let $e^{i,j}$, with $2n-1 \leq i \leq 2(n+m)$ and $-2(n+m)-1 \leq j \leq -2n$, be the $(i+j)$-cell corresponding to $e^i \wedge e^j$ in N, where e^i is the i-cell of $L^{2(n+m)}_{2n-1}$ and e^j is the j-cell of $L^{-2n}_{-2(n+m)-1}$. Note that $N^j_j = \vee S^j_\alpha$ is a wedge of some j-spheres. Let $S^{i,j}$ be the summand of $H^*(N^{(i+j)}/N^{(i+j-1)})$ corresponding to the cell $e^{i,j}$. Let a_i generate $H^i(L^{2(n+m)}_{2n-1}) \approx \mathbf{Z}_2$ for $2n-1 \leq i \leq 2(n+m)$ and b_j generate $H^i(L^{-2n}_{-2(n+m)-1}) \approx \mathbf{Z}_2$ for $-2(n+m)-1 \leq j \leq -2n$. Then $S^{i,j}$ is generated by the element $a_i \otimes b_j$. Note that both $\mathrm{Ext}_A(S^{2i,-2i-1}, \mathbf{Z}_2)$ and $\mathrm{Ext}_A(S^{2i-1,-2i}, \mathbf{Z}_2)$ can be regarded as summands of $E_2(N^{-1}_{-1})$, while $\mathrm{Ext}_A(S^{2i,-2i}, \mathbf{Z}_2)$ and $\mathrm{Ext}_A(S^{2i-1,-2i-1}, \mathbf{Z}_2)$ can be regarded as summands of $E_2(N^0_0)$ and $E_2(N^{-2}_{-2})$ respectively.

Definition 4.1. Suppose that $n \leq i \leq n+m$. Define $x^s_i \in E^{s,s}_2(N^0_0)$, $y^s_i, z^s_i \in E^{s,s-1}_2(N^{-1}_{-1})$ and $u^s_i \in E^{s,s-2}_2(N^{-2}_{-2})$ to be the classes satisfying

x^s_i generates the summand $\mathrm{Ext}^{s,s}_A(S^{2i,-2i}, \mathbf{Z}_2)$,

y^s_i generates the summand $\mathrm{Ext}^{s,s-1}_A(S^{2i,-2i-1}, \mathbf{Z}_2)$,

z_i^s generates the summand $\mathrm{Ext}_A^{s,s-1}(S^{2i-1,-2i}, \mathbf{Z}_2)$,
u_i^s generates the summand $\mathrm{Ext}_A^{s,s-2}(S^{2i-1,-2i-1}, \mathbf{Z}_2)$.

Definition 4.2. Suppose $i = 0, 1, 2$. Define sub vector spaces of $H^*(N^{-i})$ as follows:

(i) $M_i^{1,2}$ is spanned by the odd classes

$$\{a_{2j-1} \otimes b_{-2k} \mid n \le j, k \le n+m,\ 2(j-k)-1 \le -i\};$$

(ii) $M_i^{2,1}$ is spanned by the odd classes

$$\{a_{2j} \otimes b_{-2k-1} \mid n \le j, k \le n+m,\ 2(j-k)-1 \le -i\};$$

(iii) $M_i^{1,1}$ is spanned by even classes

$$\{a_{2j-1} \otimes b_{-2k-1} \mid n \le j, k \le n+m,\ 2(j-k)-2 \le -i\};$$

(iv) $M_i^{2,2}$ is spanned by the even classes

$$\{a_{2j} \otimes b_{-2k} \mid n \le j, k \le n+m,\ 2(j-k) \le -i\}.$$

Obviously these are all sub A-modules of $H^*(N^{-i})$.

There is an isomorphism $H^*(N^{-i}) \approx \bigoplus_{j,k} M_i^{j,k}$ as A-modules for $i = 0, 1, 2$. Thus

$$\mathrm{Ext}_A(H^*(N^{-i}), \mathbf{Z}_2) \approx \bigoplus_{j,k} \mathrm{Ext}_A(M_i^{j,k}, \mathbf{Z}_2). \tag{4.3}$$

The following lemma is an easy consequence of Lemma 3.2.

Lemma 4.4. *Let $M = \bigoplus_{n\ge 0} M_n$ be a graded A-module with $M_{odd} = 0$. If $\mathrm{Sq}^2 : M_0 \to M_2$ is injective, then $\mathrm{Ext}_A^{4k+3,12k+6}(M, \mathbf{Z}_2) = 0$.*

Proof. Consider the algebraic Atiyah-Hirzebruch spectral sequence converging to $\mathrm{Ext}_A(M, \mathbf{Z}_2)$ with

$$E_1^{s,t} = \bigoplus_{k\ge 0} \mathrm{Ext}_A^{s,t}(M_{2k}, \mathbf{Z}_2).$$

Then by Lemma 3.2, the possible nontrivial classes in $E_1^{4k+3,12k+6}$ come from the nontrivial classes in $\mathrm{Ext}_A^{4k+3,12k+6}(M_0, \mathbf{Z}_2)$ of filtration $4k+3$. However each class of M_0 supports a Sq^2 to M_2. This forces each class in $\mathrm{Ext}_A^{3,6}(M_0, \mathbf{Z}_2)$ corresponding to the class $h_1^3 \in \mathrm{Ext}_A^{3,6}(\mathbf{Z}_2, \mathbf{Z}_2)$ to be hit by a d_2-differential on a class in $\mathrm{Ext}_A^{2,6}(M_2, \mathbf{Z}_2)$ corresponding to the class $h_1^2 \in \mathrm{Ext}_A^{2,4}(\mathbf{Z}_2, \mathbf{Z}_2)$. Then by Adams periodicity, each nontrivial class in $\mathrm{Ext}_A^{4k+3,12k+6}(M_0, \mathbf{Z}_2)$ is hit by a d_2-differential eventually on a class in $\mathrm{Ext}_A^{4k+2,12k+6}(M_2, \mathbf{Z}_2)$ in the spectral sequence. □

Lemma 4.5. *(i) If $s \ge m+2$ and $j = 0, 1, 2$, then $E_2^{s,s-j-1}(N^{-j-2}) = 0$.*
(ii) If n is odd or $m \not\equiv 2 \pmod 4$, then x_i^s survives to $E_2^{s,s}(N^0)$ when $s \ge m$.

Proof. For (i), consider the algebraic Atiyah-Hirzebruch spectral sequence converging to $\mathrm{Ext}_A(H^*(N^{-j-2}), \mathbf{Z}_2)$ with

$$E_1^{s,t} = \bigoplus_{k=-2m-2}^{-j-2} \mathrm{Ext}_A^{s,t}(H^*(N_k^k), \mathbf{Z}_2).$$

Recall that all cells of N are of dimension $\ge -2m-2$.

Suppose $j = 1, 2$ and $s \geq m+2$. Then each summand $\mathrm{Ext}_A^{s,s-j-1}(H^*(N_k^k), \mathbf{Z}_2) = 0$ in $E_1^{s,s-j-1}$ in the spectral sequence by Lemma 3.2, hence (i) is true in this case.

Suppose $j = 0$ and $s \geq m+2$. Then $\mathrm{Ext}_A^{s,s-1}(H^*(N_k^k), \mathbf{Z}_2) = 0$ by Lemma 3.2 for

$$k \geq \begin{cases} -2m-1 \quad \text{or} \\ 2m+1 \not\equiv 3 \ (\bmod 8) \quad \text{when } k = -2m-2. \end{cases}$$

Let $2m+1 \equiv 3$ (mod 8) and $k = -2m-2$. Then the nontrivial classes in $E_1^{s,s-1}$ in this case come from $\mathrm{Ext}^{m+2,m+1}(H^*(N_{-2m-2}^{-2m-2}), \mathbf{Z}_2)$. Note that $N_{-2m-2}^{-2m-2} = S^{-2m-2}$ is the bottom cell $e^{2n-1,-2(n+m)-1}$. Consider the sub A-module $M_2^{1,1}$ of $H^*(N^{-2})$. Note that $m+2 \equiv 3$ (mod 4) and the bottom class of $M_2^{1,1}$ is of dimension $-2m-2$. By Lemma 4.4, we have $\mathrm{Ext}_A^{m+2,m+1}(M_2^{1,1}, \mathbf{Z}_2) = 0$. This implies $E_1^{m+2,m+1} = 0$ and hence (i) follows.

For (ii), consider the algebraic Atiyah-Hirzebruch spectral sequence converging to $\mathrm{Ext}_A(M_0^{2,2}, \mathbf{Z}_2)$ with

$$E_1^{s,t} = \bigoplus \mathrm{Ext}_A^{s,t}(S^{2k,-2h}, \mathbf{Z}_2)$$

where the sum runs over $n \leq k, h \leq n+m$ and $2(k-h) \leq 0$. Similarly to (i), if $s \geq m+1$, then $\mathrm{Ext}_A^{s,s-1}(S^{2k,-2h}, \mathbf{Z}_2) = 0$ for

$$\begin{cases} -2(k-h) \leq 2m-2 \quad \text{or} \\ 2m \not\equiv 4 \ (\bmod 8) \quad \text{when } -2(k-h) = 2m. \end{cases}$$

Let $-2(k-h) = 2m$ and $2m \equiv 4$ (mod 8). Then $S^{2k,-2(k+h)} = S^{2n,-2(n+m)}$. The nontrivial classes in $E_1^{s,s-1}$ for $s \geq m+1$ come from $\mathrm{Ext}_A^{m+1,m}(S^{2n,-2(n+m)}, \mathbf{Z}_2)$. Now applying Lemma 4.4 again, we have $\mathrm{Ext}_A^{m+1,m}(M_0^{2,2}) = 0$; hence $\mathrm{Ext}_A^{s,s-1}(M_0^{2,2}) = 0$ if $s \geq m+1$. This implies that $x_i^s \in \mathrm{Ext}_A^{s,s}(S^{2i,-2i}, \mathbf{Z}_2)$ survives to a nontrivial element of $\mathrm{Ext}_A^{s,s}(M_0^{2,2}, \mathbf{Z}_2)$ if $s \geq m$, and (ii) follows from the fact that

$$\mathrm{Ext}_A(H^*(N^0), \mathbf{Z}_2) \approx \bigoplus_{a,b} \mathrm{Ext}_A(M_0^{a,b}, \mathbf{Z}_2)$$

by (4.3). □

Remark 4.6. (i) By Lemma 4.5 (*ii*), x_i^s can be regarded as a class in $E_2^{s,s}(N^0)$ if $s \geq m$. By applying Lemma 4.5 (*i*) to the exact sequence

$$\to E_2^{s,s-1}(N^{-1}) \to E_2^{s,s-1}(N_{-2}^{-1}) \to E^{s+1,s-1}(N^{-3}),$$

we see that both y_i^s and z_i^s can also be regarded as classes in $E_2^{s,s-1}(N^{-1})$ if $s \geq m+1$. Similarly u_i^s can be regarded as a class of $E_2^{s,s-2}(N^{-2})$ if $s \geq m+1$.

(ii) The set $\{x_i^s\}$ generates $E_2^{s,s}(N^0)$ if $s \geq m$, while the sets $\{y_i^s, z_i^s\}$ and $\{u_i^s\}$ generate $E_2^{s,s-1}(N^{-1})$ and $E_2^{s,s-2}(N^{-2})$ respectively if $s \geq m+1$.

Lemma 4.7. (*i*) *If $s \geq m+1$, then both y_i^s and z_i^s survive to $E_v^{s,s-1}(N^{-1})$, while x_i^s and u_i^s survive to $E_v^{s,s}(N^0)$ and $E_v^{s,s-2}(N^{-2})$ respectively.*

(*ii*) *Suppose either n is odd or $m \not\equiv 2$ (mod 4). If $s \geq m$, then x_i^s survives to $E_v^{s,s}(N^0)$.*

Proof. We just prove (i). The proof for (ii) is similar.

If $v = 2$, it is true already by Remark 4.6. Assume $v \geq 3$. Consider the differential d_r with $2 \leq r \leq v-1$. For $r=2$ and $j=0,1,2$, the projection

$$p_* : E_2^{s,s-j-1}(N^{-j}) \to E_2^{s,s-j-1}(N^{-j}_{-j-1}) \tag{4.8}$$

is injective when $s \geq m+2$ by Lemma 4.5 (i) and exactness. Then applying Lemma 3.1, we see that the differential d_r is null on $p_*(x_i^s), p_*(y_i^s), p_*(z_i^s)$ and $p_*(u_i^s)$ in $E_2(N^{-j}_{-j-1})$ when $s \geq m+1$. Thus $d_r(w) = 0$ in $E_2(N^{-j})$ by (4.8) for $w = x_i^s, y_i^s, z_i^s$ and u_i^s, and (i) is true if $v = 3$. If $v \geq 4$, then (4.8) implies that for $j = 0,1,2$,

$$p_* : E_3^{s,s-j-1}(N^{-j}) \to E_3^{s,s-j-1}(N^{-j}_{-j-1}) \tag{4.9}$$

is injective when $s \geq m+2$ because of the triviality of d_2 on $E_2^{s-r,s-r-j}(N^{-j}_{-j-1})$ by Lemma 3.1. Then repeating the same process as before, we see that d_3 is null on x_i^s, y_i^s, z_i^s and u_i^s when $s \geq m+1$. Continuing this process, we obtain a monomorphism

$$E_r^{s,s-j-1}(N^{-j}) \to E_r^{s,s-j-1}(N^{-j}_{-j-1})$$

for $2 \leq r \leq v-1$ and $j = 0,1,2$; and eventually d_r is null on x_i^s, y_i^s, z_i^s and u_i^s when $s \geq m+1$. □

The proof above also implies that $E_v^{s,s-j-1}(N^{-j}) \to E_v^{s,s-j-1}(N^{-j}_{-j-1})$ is injective for $j = 0,1,2$. Projecting N^{-j} to N^{-j}_{-j-1} and then applying Lemma 3.1, we have (i) and (ii) in the following proposition, while (iii) is derived from (i) and (ii). Here note that, by the injectivity of the projection

$$p_* : E_r^{s,s-1}(N^0) \to E_r^{s,s-1}(N^0_{-1})$$

for $2 \leq r \leq v-1$ and $s \geq m+1$, we see that $\{y_i^s, z_i^s\}$ are not hit by differentials in $E_r(N^0)$ if $s \geq m+1$, hence survive also to $E_v^{s,s-1}(N^0)$. Actually $\{y_i^s, z_i^s\}$ generate $E_v^{s,s-1}(N^0)$ if $s \geq m+1$, which implies that $\{u_i^s\}$ survive also to $E_v^{s,s-2}(N^0)$ if $s \geq m+1$.

Proposition 4.10. *(i) If $s \geq m+1$, $d_v(x_i^s) = y_i^{s+v} + z_i^{s+v}$ in $E_v^{s+v,s+v-1}(N^0)$. The same formula holds when $s = m$ if either n is odd or $2m \not\equiv 4$ mod 8.*

(ii) $d_v(y_i^s) = u_i^{s+v} = d_v(z_i^s)$ in $E_v^{s+v,s+v-2}(N^{-1})$ when $s \geq m+1$.

(iii) If $s \geq m+v+1$, then $E_{v+1}^{s,s-1}(N^0) = 0$. If either n is odd or $2m \not\equiv 4$ (mod 8), then $E_{v+1}^{s,s-1}(N^0) = 0$ for $s = m+v$.

Proof of Theorem 1.1. First by Proposition 2.1 (ii) and Theorem 2.3 (iii), we have

$$|L_{2n-1}^{2(n+m)}| \geq \begin{cases} 2^{v+m} & \text{if either } n \text{ is odd or } m \not\equiv 2 \text{ (mod 4)}, \\ 2^{v+m+1} & \text{if } n \text{ is even and } m \equiv 2 \text{ (mod 4)}. \end{cases}$$

Next by Proposition 4.10 (iii), we have

$$|L_{2n-1}^{2n+m)}| \leq \begin{cases} 2^{v+m} & \text{if either } n \text{ is odd or } m \not\equiv 2 \text{ (mod 4)}, \\ 2^{v+m+1} & \text{if } n \text{ is even and } m \equiv 2 \text{ (mod 4)}. \end{cases}$$

□

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