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A CONVERSE OF THE GELFAND THEOREM

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ABSTRACT. In this short note we obtain a converse to the Gelfand theorem: a Riemannian manifold is homogeneous if the isometrically invariant operators on the manifold form a commutative algebra.

1. INTRODUCTION

A remarkable property of a symmetric space is that the isometrically invariant differential operators on it form a commutative algebra. This is known as the Gelfand theorem ([4]). The converse of the theorem is not true in general. There are many non-symmetric homogeneous Riemannian manifolds on which the algebra of isometrically invariant differential operators is commutative ([6], [11]). However, there still are many results relating to the converse of the Gelfand theorem (See [1], [7], [8], etc.). Among them is Kowalski and Vanhecke's theorem which asserts that a homogeneous Kähler manifold is locally Hermitian symmetric if all isometrically invariant differential operators on the manifold commute each other.

In this short note, we prove a converse of the Gelfand theorem:

Theorem. *If the algebra of isometrically invariant differential operators on a complete connected Riemannian manifold M is commutative, M is a Riemannian homogeneous manifold.*

A Riemannian homogeneous manifold with commutative invariant differential operators is called a commutative space ([10]). Our result says that the "homogeneous" assumption can be removed from the above definition. Commutative spaces are contained in the class of Riemannian manifolds with volume-preserving local geodesic symmetries ([9]). It is known that a naturally reductive homogeneous space of dimension ≤ 5 is necessarily a commutative space ([10]). This is no longer true for dimension ≥ 6 ([5]). On the other hand, a commutative space is not necessarily a naturally reductive homogeneous space due to the results of Kaplan and Ricci's on Heisenberg type Lie groups ([6], [11]). It is an interesting problem to characterize commutative spaces.

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2. Proof of the Theorem

In this section, M denotes a complete connected Riemannian manifold. Denote by $C^\infty(M)$ the Frèchet space of smooth functions on M with the topology under which a sequence of functions is convergent if for any integer $k \geq 0$, the sequence of the k-th derivatives of the functions is uniformly convergent in any compact subset of M. As usual, a differential operator on M is a continuous linear operator $A : C^\infty(M) \to C^\infty(M)$ which has local form $A = \sum_{|\alpha| \leq k} f_\alpha D^\alpha$. A differential operator A is called isometrically invariant if $g^* A = A$ holds for any isometry g of M, where $g^* A$ is defined by $g^* A(u)(x) = A(u \circ g)(g^{-1}x)$. It is easy to see that the Laplacian $\triangle$ on M is an isometrically invariant differential operator. To list other examples of the invariant operators, we need the following definition of the spherical mean operator.

The spherical mean operator $L_r : C^\infty(M) \to C^\infty(M)$ is defined by

$$L_r f(x) = \int_{S_x(M)} f(\exp_x r\xi) d\mu_x(\xi)$$

where $d\mu_x$ stands for the normalized canonical measure on the unit tangent sphere $S_x(M) = \{\xi \in T_x M \mid \parallel \xi \parallel = 1\}$. We refer to [3] as a reference for the operator L_r. It is easy to see that the spherical mean operator L_r is an isometrically invariant operator. We notice that L_r is smooth in r. Let Z be the vector field on $S(M)$ generating the geodesic flow G^t. A direct calculation shows that for $k = 1, 2, \cdots$,

$$\frac{d^k}{dr^k}|_{r=0} L_r f(x) = \int_{S_x(M)} (Z^k \pi^* f)(\xi) d\mu_x(\xi),$$
$$Z^k(\pi^* f)(\xi) = \bigtriangledown^k_{\xi \ldots \xi} f = \sum_{\alpha_1 \ldots \alpha_k} \xi^{\alpha_1} \ldots \xi^{\alpha_k} \bigtriangledown_{\alpha_k} \cdots \bigtriangledown_{\alpha_1} f,$$

where $\pi : S(M) \to M$ is the natural projection, $\xi = \sum \xi^i \frac{\partial}{\partial x^i} \in S_x(M)$, and $\bigtriangledown$ is the covariant differential on M. Applying the above identity to the Taylor expansion of L_r, we get that, for each integer $n \geq 0$,

$$L_r = \sum_{k \leq n} \frac{1}{(2k)!} P_{2k} r^{2k} + \frac{d^{2n+1}}{dr^{2n+1}}|_{r=\bar{r}} (L_r) \frac{r^{2n+1}}{(2n+1)!}$$

where

$$P_{2k} f(x) = \int_{S_x(M)} \sum \xi^{\alpha_1} \ldots \xi^{\alpha_{2k}} \bigtriangledown_{\alpha_{2k}} \cdots \bigtriangledown_{\alpha_1} (f)(x) d\mu_x(\xi).$$

Since L_r is an isometrically invariant operator, so is each coefficient P_{2k}.

If the algebra of isometrically invariant differential operators on M is commutative, M has cyclically parallel Ricci tensor by a theorem of Z. Szabo's ([12]). Therefore, M is real analytic (see [2], [12]). In this case, for a local analytic function f,

$$L_r f(x) = \sum_k \frac{1}{(2k)!} P_{2k}(f)(x) r^{2k}$$

holds for small enough $r > 0$ such that $\exp_x(r\xi)$ is contained in the analytic domain of f. Since analytic functions are locally dense in $C^\infty(M)$, we get the following lemma:

Lemma 1. *If M is a Riemannian manifold with commutative algebra of isometrically invariant differential operators, then the spherical mean operator L_r commutes with all isometrically invariant differential operators.*

In the rest of the paper, we denote by G the isometry group of M. To prove the theorem, we need the following lemma:

Lemma 2. *Let $x \in M$ and $U \subseteq M$ be an open set with compact closure $\bar{U}$. The set $K_x = \{g \in G \mid gx \in \bar{U}\}$ is pre-compact in G with regard to the compact-open topology.*

Proof. Let $\{g_n\}_1^\infty \subseteq K_x$ be a sequence. According to the definition of K_x, we see that $\{g_n x\} \subseteq \bar{U}$. There is a subsequence $\{g_{n_i}\}$ of $\{g_n\}$, due to the compactness of $\bar{U}$, such that $\{g_{n_i}(x)\}$ converges to a point $x_1 \in M$. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame of M at x. Since $(g_{n_i})_*(e_j)$ are contained in a compact subset of the sphere tangent bundle $S(M)$, by passing to a subsequence if necessary, we may assume that $\lim_{i\to\infty}(g_{n_i})_*(e_j)$ exists for all $j = 1, \dots, n$. Therefore, $\{(g_{n_i})_*(e)\}$ is uniformly convergent in $S_x M$.

We will show that $\{g_{n_i}(p)\}$ is uniformly convergent for p in any compact set of M. To this end, choose a geodesic $\gamma : [0, l] \to M$ connecting x with p. Since g_{n_i} are isometries, $\varphi_{n_i}(t) = g_{n_i}(\gamma(t))$ are also geodesics with initial data $\varphi_{n_i}(0) = g_{n_i}(x), \varphi'_{n_i}(0) = (g_{n_i})_*\gamma'(0)$. Hence, by the convergence of $g_{n_i}(x)$ and $(g_{n_i})_*\gamma'(0)$ and the continuous dependence of ordinary differential equations on the initial data, we see that $g_{n_i}(p) = \varphi_{n_i}(l)$ is convergent; more precisely, $g_{n_i}(p)$ is uniformly convergent about p in any compact set.

Define $g(p) = \lim_{i\to\infty} g_{n_i}(p)$. It is easy to see that g is an isometry of M and $\lim g_{n_i} = g$ under the compact-open topology of G. This proves our lemma. □

Proof of Theorem. For each $y \in M$, the orbit $G(y)$ is a smooth submanifold of M. Since $G(y)$ is a closed subset of M (see [13]), it suffices to prove $\dim(G(y)) = \dim(M)$ for each $y \in M$.

Assume that $\dim(G(y)) < \dim(M)$. Take an open neighborhood U of y such that $dist^2(x, G(y))$ is smooth in U and U has compact closure $\bar{U}$. Let $\lambda(x)$ be a smooth cut-off function of U and define

$$h(x) = \lambda(x) dist^2(x, G(y)).$$

It is easy to see that $h(x)$ is a global smooth function satisfying $h(x) > 0$ for each $x \in U \setminus G(y)$ and $\mathrm{supp}(h) \subseteq \bar{U}$. Furthermore, we define

$$h_1(x) = \int_G h(gx) d\mu(g).$$

Here $d\mu$ is the Harr measure on G. Since $h(gx) \neq 0$ implies that $g \in K_x = \{g \in G \mid gx \in \bar{U}\}$, according to Lemma 2, we see that the integrand $g \to h(gx)$ has compact support in G. Therefore, the function h_1 is well-defined. Moreover, h_1 is a non-negative G-invariant function and satisfies $h_1 > 0$ in $U \setminus G(y)$.

Obviously, $h_1 \triangle$ is a G-invariant differential operator. Here $\triangle$ is the Laplacian on M. According to Lemma 1, we get $[h_1\triangle, L_r] = 0$. On the other hand, we still have $[\triangle, L_r] = 0$. Therefore, it holds that $h_1 L_r \triangle = L_r(h_1\triangle)$. Take $f \in C^\infty(M)$ such that $\triangle f = 1$ in U. Then for $r > 0$ small enough such that $\exp_y(r\xi) \in U$ for

$\xi \in S_y(M)$, we have $h_1(y)L_r(1) = L_r(h_1)$, i.e.,

$$h_1(y) = \int_{S_y(M)} h_1(\exp_y(r\xi))d\mu(\xi).$$

However $h_1(y) = 0$ and $\int_{S_y(M)} h_1(\exp_y(r\xi))d\mu(\xi) > 0$. This is a contradiction. □

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