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# THE INFINITESIMAL CONE OF A TOTALLY POSITIVE SEMIGROUP

KONSTANZE RIETSCH

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Abstract. Given a complex reductive linear algebraic group split over $\mathbb{R}$ with a fixed pinning, it is shown that all elements of the Lie algebra $\mathfrak{g}$ infinitesimal to the totally positive subsemigroup $G_{\geq 0}$ of $G$ lie in the totally positive cone $\mathfrak{g}_{\geq 0} \subset \mathfrak{g}$.

## 1. Introduction

Classically, the totally positive subsemigroup of $GL_n\mathbb{R}$ is the semigroup consisting of the matrices in $GL_n\mathbb{R}$ all of whose minors are nonnegative. Corresponding to it, there is a cone in $\mathfrak{gl}_n\mathbb{R}$, the "infinitesimal semigroup", studied by Loewner in [Loe]. It consists of all matrices with arbitrary entries along the diagonal, nonnegative entries next to the diagonal, and zeros everywhere else. Loewner showed that $X$ lies in this cone if and only if $\exp(tX)$ lies in the totally positive semigroup for all $t \geq 0$. Loewner also pointed out that, as a consequence of a theorem of Whitney's, the exponential image of the infinitesimal semigroup generates the totally positive semigroup.

Let $G$ be an arbitrary complex reductive linear algebraic group split over $\mathbb{R}$. We will start by recalling Lusztig's definitions of the totally $\geq 0$ subsemigroup $G_{\geq 0}$ of $G$, and of a cone $\mathfrak{g}_{\geq 0}$ in $\mathfrak{g} = \mathrm{Lie}(G)$. These definitions give both $G_{\geq 0}$ and $\mathfrak{g}_{\geq 0}$ in terms of a set of generators. The aim of this paper is to extend Loewner's result relating $G_{\geq 0}$ and $\mathfrak{g}_{\geq 0}$ to this more general setting. This is partially done in [L1] where it is shown that if $X \in \mathfrak{g}_{\geq 0}$ then $\exp X$ lies in $G_{\geq 0}$. To complete the picture we show (Proposition 2.1) that if $\exp(tX)$ lies in $G_{\geq 0}$ for all $t \geq 0$, then $X$ must lie in $\mathfrak{g}_{\geq 0}$.

The proof of Proposition 2.1 in the classical case ([Loe]) uses the definition of $G_{\geq 0}$ in terms of inequalities, rather than by generators. For arbitrary semisimple, simply laced $G$ there is also a characterization of $G_{\geq 0}$ by inequalities, arising from the positivity property of Lusztig's canonical basis (see Theorem 3.1, shown by Lusztig in [L1]; for a stronger version, see [L2]). We make use of these inequalities to prove Proposition 2.1 in the simple, simply laced case. The general case then follows by standard arguments.

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## 2. Definitions and proposition

Let $G$ be a complex reductive linear algebraic group split over $\mathbb{R}$. We will identify any algebraic group over $\mathbb{R}$ with its $\mathbb{R}$-rational points. Fix a maximal torus $T \subset G$ that is split over $\mathbb{R}$, and let $B^+$ be a Borel subgroup containing $T$. Denote the set of simple roots corresponding to $B^+$ by $\Pi = \{\alpha_i | i \in I\}$, and the set of (positive) roots by $R$ (resp. $R^+$). Let $X^\vee(T)$ denote the group of cocharacters $\chi : \mathbb{R}^* \to T$. For any root $\alpha$, denote its coroot by $\alpha^\vee$. Let $(T, B^+, B^-, x_i, y_i; i \in I)$ be a *pinning* of $G$ (see [L1]). Here $B^-$ is a Borel subgroup containing $T$ opposed to $B^+$. Let $U^+$, $U^-$ be the unipotent radicals of $B^+$, resp. $B^-$. Then the $x_i$, resp. $y_i$, are embeddings $x_i : \mathbb{R} \to U^+$ and $y_i : \mathbb{R} \to U^-$ such that

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mapsto x_i(a), \qquad \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mapsto y_i(b), \qquad \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \mapsto \alpha_i^\vee(c)$$

defines a homomorphism $SL_2\mathbb{R} \to G_i = \mathrm{Centralizer}(\mathrm{Ker}(\alpha_i)^\circ)$.

The totally $\geq 0$ submonoid $G_{\geq 0}$ of $G$ is then defined (in [L1, p.535]) to be the semigroup with identity element generated by the elements $\{x_i(a), y_i(a), \chi(b) | a \in \mathbb{R}_{\geq 0}, b \in \mathbb{R}_{>0}, i \in I, \chi \in X^\vee(T)\}$.

Let $dx_i$, $dy_i$ be the derivatives of $x_i$, $y_i$ at zero. Lusztig defines in [L1, p. 549] a cone in $\mathfrak{g} := \mathrm{Lie}(G)$ as follows:

$$\mathfrak{g}_{\geq 0} := \{t + \sum_{i \in I} (a_i dx_i + b_i dy_i) |\ t \in \mathrm{Lie}(T),\ a_i, b_i \geq 0\}.$$

We propose to show

**Proposition 2.1.** *If $X \in \mathfrak{g}$ satisfies*

$$(*) \qquad \exp(tX) \in G_{\geq 0} \quad \textit{for} \quad t \geq 0$$

*then $X$ lies in $\mathfrak{g}_{\geq 0}$.*

The converse, $X \in \mathfrak{g}_{\geq 0}$ implies (*), was shown by Lusztig in [L1].

## 3. Connections with Lusztig's canonical basis

In the following assume $G$ semisimple and of simply laced type. Let $\mathbb{U}$ denote the universal enveloping algebra of $\mathfrak{g}$. Corresponding to the chosen pinning there is a set of Chevalley generators of $\mathbb{U}$ given by $E_i := dx_i$ and $F_i := dy_i$ (here $\mathfrak{g}$ is considered a subset of $\mathbb{U}$). Let $\mathbb{U}^-$ be the subalgebra of $\mathbb{U}$ generated by the elements $F_i$ ($i \in I$). Denote by $\mathbb{B}^-$ the basis of $\mathbb{U}^-$ obtained from Lusztig's canonical basis [L0] of the corresponding quantized universal enveloping algebra by specializing at $v = 1$ and extending scalars from $\mathbb{Q}$ to $\mathbb{C}$.

Consider a finite dimensional, irreducible, algebraic representation $V$ of $G$ over $\mathbb{C}$ and the corresponding representation of $\mathbb{U}$ on $V$. Let $B(\eta)$ denote Lusztig's canonical basis of $V$ arising from applying the elements of $\mathbb{B}^-$ to a chosen highest weight vector $\eta$. In relation to $G_{\geq 0}$, the basis $B(\eta)$ has the following positivity property (see [L1, Prop 3.2]).

**Theorem 3.1**(Lusztig). *Let $g \in G_{\geq 0}$. Then the matrix entries of $g : V \to V$ with respect to $B(\eta)$ are nonnegative real numbers.*

This property will be central to our proof of Proposition 2.1. It has the following consequence.

**Lemma 3.2.** *Suppose $X$ satisfies property (*) of Proposition 2.1 and $B$ is a basis of $V$ such that $G_{\geq 0}$ acts on $V$ by matrices with nonnegative real entries with respect to $B$. Then for any $b, b' \in B$ with $b \neq b'$, the coefficient of $b'$ in $X.b$ is nonnegative.*

*Proof.* Let $\langle , \rangle$ be an inner product making $B$ orthonormal. By our assumptions, we have

$$\langle \exp(tX).b, b' \rangle \geq 0 \qquad \forall\ t \geq 0.$$

At $t = 0$ the above becomes an equality ( note that $b \neq b'$ ). Therefore taking the derivative at zero gives

$$\langle X.b, b' \rangle \geq 0.$$

□

3.3. **Canonical basis of the adjoint representation.** Suppose $G$ is simple. We consider the adjoint representation $V = \mathfrak{g}$. Let $B = \{X_\alpha, t_i | \alpha \in R, i \in I\}$ be a basis of $V$ such that, as in [L3, p 259],

$$E_i.X_\alpha = \begin{cases} X_{\alpha+\alpha_i}, & \alpha + \alpha_i \in R, \\ t_i, & \alpha + \alpha_i = 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$F_i.X_\alpha = \begin{cases} X_{\alpha-\alpha_i}, & \alpha - \alpha_i \in R, \\ t_i, & \alpha - \alpha_i = 0, \\ 0, & \text{otherwise}, \end{cases}$$

$$E_i.t_j = \begin{cases} -\langle \alpha_i, \alpha_j{}^\vee \rangle X_{\alpha_i}, & i \neq j, \\ 2X_{\alpha_i}, & i = j, \end{cases}$$

$$F_i.t_j = \begin{cases} -\langle \alpha_i, \alpha_j{}^\vee \rangle X_{-\alpha_i}, & i \neq j, \\ 2X_{-\alpha_i}, & i = j. \end{cases}$$

Lusztig has pointed out that $B$ coincides with the canonical basis $B(X_{\alpha_0})$ of $\mathfrak{g}$ corresponding to highest weight vector $X_{\alpha_0}$ (if $\alpha_0$ is the highest root). This can be proved as follows. To check that $X_\alpha$ lies in $B(X_{\alpha_0})$ note that $\alpha$ is conjugate to $\alpha_0$ by some element of the Weyl group. Then [L4, 28.1.4] gives an explicit formula for the canonical basis element of weight $\alpha$ and it is seen to coincide with $X_\alpha$. To show that the $t_i$ lie in $B(X_{\alpha_0})$ pick an element $b$ of the canonical basis of $\mathbb{U}^-$ such that $X_{\alpha_i} = b.X_{\alpha_0}$. It follows that $b \notin F_i.\mathbb{U}^-$. Now applying [L4, 14.3.2(c)] and the positivity property of the canonical basis of $\mathbb{U}^-$ we find that

$$F_i.b = b' + \text{elements in } F_i^2.\mathbb{U}^-,$$

where $b' \in \mathbb{U}^-$ is again an element of the canonical basis. Then we see that

$$t_i = F_i.X_{\alpha_i} = F_i.b.X_{\alpha_0} = b'.X_{\alpha_0} + 0.$$

Therefore $t_i$ lies in $B(X_{\alpha_0})$ and the proof is complete.

We note that, following [L1, Proof of Proposition 3.2], one can also show directly that $B$ has the property of Theorem 3.1, i.e. that for all $g \in G_{\geq 0}$ the matrix entries of $g : V \to V$ with respect to $B$ lie in $\mathbb{R}_{\geq 0}$. In fact, this property need only be checked for the generators $\exp(aE_i), \exp(aF_i)$, and $\chi(b)$ ( $a \in \mathbb{R}_{\geq 0},\ b \in \mathbb{R}_{>0}$ ) of $G_{\geq 0}$, where it can be deduced from the observation that the $E_i$ and $F_i$ act by matrices with nonnegative entries with respect to $B$.

## 4. Proof of Proposition 2.1

Assume first that $G$ is simple, as above, and consider the basis of the adjoint representation from 3.3. We choose an element $i_0 \in I$ and rescale the basis such that $X_{\alpha_{i_0}} = dx_{i_0}$. The resulting basis is the canonical basis for the rescaled highest weight vector and is again denoted by $B$.

**Definition and Lemma 4.1.** *For $i \in I$ define $\sigma_i \in \mathbb{R}$ by $X_{\alpha_i} = \sigma_i dx_i$. Then*

(1) $\sigma_i = \pm 1$,
(2) $\sigma_i$ *alternates along edges of the Dynkin diagram.*

*Proof.* If $i$ and $j \in I$ are connected by an edge, then

$$[X_{\alpha_i}, X_{\alpha_j}] = [\sigma_i dx_i, X_{\alpha_j}] = \sigma_i X_{\alpha_i+\alpha_j},$$

wherefore

$$[X_{\alpha_j}, X_{\alpha_i}] = -\sigma_i X_{\alpha_i+\alpha_j} = -\sigma_i [dx_j, X_{\alpha_i}].$$

This implies $X_{\alpha_j} = -\sigma_i dx_j$. So $\sigma_j = -\sigma_i$, and (2) follows. Since in every connected component of the Dynkin diagram there is an $i_0$ with $\sigma_{i_0} = 1$, (1) also follows. $\square$

*Remark* 4.2. We recall the following two facts about finite type, simply laced root systems:

(1) Let $\alpha \in R^+$. Either $\alpha$ is simple or there are a positive root $\alpha'$ and a simple reflection $S_{\alpha_i}$ such that $S_{\alpha_i}\alpha' = \alpha' + \alpha_i = \alpha$.
(2) For any $\lambda = \sum_{i\in I} \lambda_i \alpha_i \in R$, the set of vertices $\mathrm{Supp}(\lambda) := \{i \in I | \lambda_i \neq 0\}$ is connected in the Dynkin diagram.

**4.3 Lemma.** *Let $\alpha = \sum_{j\in I} n_j \alpha_j$ be a positive root of height $m$. Then*

$$[X_\alpha, X_{-\alpha}] = (-1)^{m+1} \sum_{j\in I} \sigma_j n_j t_j. \tag{**}$$

*Proof.* If $ht(\alpha) = 1$, say $\alpha = \alpha_i$, then

$$[X_\alpha, X_{-\alpha}] = \sigma_i E_i.X_{-\alpha_i} = \sigma_i t_i.$$

We prove the general case by induction. Let $ht(\alpha) = m \geq 2$. Using (4.2.1), pick $\alpha' \in R^+$ such that $\alpha = \alpha' + \alpha_i = S_{\alpha_i}\alpha$ for some $i \in I$. Then

$$\begin{aligned}[X_\alpha, X_{-\alpha}] &= [E_i.X_{\alpha'}, X_{-\alpha}] = E_i.[X_{\alpha'}, X_{-\alpha}] - [X_{\alpha'}, X_{-\alpha'}]\\ &= E_i.[X_{\alpha'}, F_i.X_{-\alpha'}] - [X_{\alpha'}, X_{-\alpha'}] = (E_iF_i - 1).[X_{\alpha'}, X_{-\alpha'}].\end{aligned}$$

Denote by $I_i$ the set of vertices $j \in I \setminus \{i\}$ adjacent to $i$ in the Dynkin diagram. If $\alpha' = \sum_j n'_j \alpha_j$, then

$$n_i = \left(\sum_{j\in I_i} n'_j\right) - n'_i, \qquad \text{and} \qquad n_j = n'_j \qquad (j \neq i).$$

Now (**) holds for $\alpha'$, by induction hypothesis. So, using the above identities and Lemma 4.1,

$$\begin{aligned}
[X_\alpha, X_{-\alpha}] &= (E_iF_i - 1).(-1)^m \sum_{j\in I} \sigma_j n'_j t_j \\
&= (-1)^m \sigma_i n'_i t_i + (-1)^m \sum_{j\in I_i} \sigma_j n'_j (t_i - t_j) + (-1)^m \sum_{j\notin I_i \cup \{i\}} \sigma_j n'_j (0 - t_j) \\
&= \Big((-1)^m \sigma_i n'_i + (-1)^m \sum_{j\in I_i} \sigma_j n'_j\Big) t_i - (-1)^m \sum_{j\neq i} \sigma_j n'_j t_j \\
&= (-1)^m \sigma_i \Big(n'_i - \sum_{j\in I_i} n'_j\Big) t_i + (-1)^{m+1} \sum_{j\neq i} \sigma_j n'_j t_j \\
&= (-1)^{m+1} \sum_{j\in I} \sigma_j n_j t_j.
\end{aligned}$$

□

**4.4 Proof of Proposition 2.1.** Let $X \in \mathfrak{g}$ satisfy (*). Write

$$X = \sum_{i\in I} \lambda_i dx_i + \sum_{i\in I} \lambda_{-i} dy_i + \sum_{\alpha\in R\setminus\pm\Pi} \mu_\alpha X_\alpha + t, \qquad t \in \mathrm{Lie}(T).$$

We must show that $\lambda_{\pm i} \geq 0$ and $\mu_\alpha = 0$. To do this we use the positivity properties of $B$ established in 3.3.

(1) $[X, X_{-\alpha_i}]$ has Lie($T$)-component $\lambda_i t_i$. Therefore, by Lemma 3.2, we have $\lambda_i \geq 0$. Similarly, considering the Lie($T$)-component of $[X, X_{\alpha_i}]$ gives that $\lambda_{-i} \geq 0$.

(2) Write $\alpha = \sum_j n_j \alpha_j$. Consider $[X, X_{-\alpha}]$. Its Lie($T$)-component is

$$\mu_\alpha [X_\alpha, X_{-\alpha}] = \varepsilon \mu_\alpha \sum_j \sigma_j n_j t_j,$$

where $\varepsilon = \pm 1$ depending on the height of $\alpha$ (Lemma 4.3). Choose vertices $i, j \in \mathrm{Supp}(\alpha)$ adjacent in the Dynkin diagram (4.2.2). By Lemma 3.2, $\varepsilon\mu_\alpha\sigma_j n_j \geq 0$ and $\varepsilon\mu_\alpha\sigma_i n_i \geq 0$. Since $\sigma_i = -\sigma_j$ (Lemma 4.1), this implies $\mu_\alpha = 0$.

This proves the propsition in the simple, simply laced case. The more general assertion of the proposition can be reduced to this special case by standard arguments.

*Remark 4.5.* Proposition 2.1 can also be proved using the theory of Lie semigroups. Let $U^+_{\geq 0}$ and $U^-_{\geq 0}$ be the subsemigroups of $U^+$, resp. $U^-$, generated by $\{x_i(a) | a \in \mathbb{R}_{\geq 0},\ i \in I\}$, resp. by $\{y_i(a) | a \in \mathbb{R}_{\geq 0},\ i \in I\}$. Correspondingly define $\mathfrak{u}^+_{\geq 0} = \sum_{i\in I} \mathbb{R}^+ dx_i$ in $\mathfrak{u}^+$, and $\mathfrak{u}^-_{\geq 0} = \sum_{i\in I} \mathbb{R}^+ dy_i$ in $\mathfrak{u}^-$. Then one can check that the conditions for [HN, Prop. 1.43] with $G = U^\pm$, $W = \mathfrak{u}^\pm_{\geq 0}$ and $N = (U^\pm, U^\pm)$, are satisfied. From that proposition it follows that $\mathfrak{u}^\pm_{\geq 0}$ is the tangent cone to $U^\pm_{\geq 0}$. This result together with [L1, Lemma 2.3(b)] suffices to prove that $\mathfrak{g}_{\geq 0}$ is the tangent cone of $G_{\geq 0}$.

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Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

*E-mail address*: `rietsch@math.mit.edu`