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A CHARACTERIZATION OF RINGS IN WHICH EACH PARTIAL ORDER IS CONTAINED IN A TOTAL ORDER

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ABSTRACT. Rings in which each partial order can be extended to a total order are called O^*- rings by Fuchs. We characterize O^*- rings as subrings of algebras over the rationals that arise by freely adjoining an identity or one-sided identity to a rational vector space N or by taking the direct sum of N with an O^*-field. Each real quadratic extension of the rationals is an O^*- field.

A ring R is called an O^*- ring if each of its ring partial orders can be extended to a total order of R. Two of the problems in the list at the back of Fuch's book [4] concern O^*- rings.

(A) Establish ring theoretical properties of O^*- rings.

(B) Does there exist a polynomial identity which forces each totally ordered ring that satisfies it to be an O^*- ring?

These problems were perhaps motivated by the well-known fact that each torsion-free abelian group is an O^*- group. Recently, Kreinovich [7] has shown that (B) has a negative answer in the sense that if $f(x_1, \cdots, x_n) = 0$ is such an identity, then each O^*- ring that satisfies it must be trivial; that is, $R^2 = 0$. In the process of showing this he noted that an O^*- ring has two very restrictive properties: it is algebraic over $\mathbb{Z}$ and each nilpotent element has index at most two. To see this first recall that the partial order in a partially ordered ring R is determined by its positive cone $R^+ = \{x \in R : x \geq 0\}$; we will refer to such a positive cone as a partial order of R. Now if a is an element of an O^*- ring R that is not algebraic over $\mathbb{Z}$, then $\mathbb{Z}^+[-a^2]$ is a partial order of R which is not contained in any total order of R. Also, if $a \in R$ is nilpotent of index $n > 2$ let $b = -a^{n-2}$ if n is even and let $b = -a^{n-1}$ if n is odd. Then $\mathbb{Z}^+b$ is a partial order of R that is not contained in any total order of R.

Clearly, each subring of an O^*- ring R is an O^*- ring, and its divisible hull $d(R)$ is also an O^*- ring. For if P is a partial order of $d(R)$ and T is a total order of R which contains $P \cap R$, then $d(T) = \{x \in d(R) : \exists n > 0 \text{ with } nx \in T\}$ is a total order of $d(R)$ which contains P. Consequently, in this paper we will deal exclusively with algebras over the rationals $\mathbb{Q}$. All such O^*- algebras are determined in the

Theorem. *If R is an O^*- algebra, then there is a $\mathbb{Q}$ - vector space N such that R is (isomorphic to) one of the following algebras.*

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(i) *$R = F \oplus N$ (algebra direct sum) where F is a subfield of the reals that is algebraic over $\mathbb{Q}$ and $N^2 = 0$.*

(ii) $R = \begin{pmatrix} \mathbb{Q} & N \\ 0 & 0 \end{pmatrix}$ *or the dual* $\begin{pmatrix} \mathbb{Q} & 0 \\ N & 0 \end{pmatrix}$.

(iii) $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a \in \mathbb{Q}\ ,\ b \in N \right\}$.

Moreover, each of these algebras is an O^*- algebra where in (i) *F is an O^*-field.*

Proof. If G and H are po-groups with positive cones P_G and P_H respectively, then $G \underset{\leftarrow}{\oplus} H$ will denote the po-group whose underlying group is the direct sum $G \oplus H$ and whose positive cone is $\{g + h : 0 \neq h \in P_H,\ \text{ or }\ h = 0\ \text{ and }\ g \in P_G\}$; and $G \underset{\rightarrow}{\oplus} H$ will denote the same group ordered similarly but with G dominating. The same arrow notation will be used for other lexicographic orderings even if there are more than two summands.

In a totally ordered ring the set N of nilpotent elements is an ideal and the quotient modulo N is a totally ordered domain [4, p.130]. Assume that $R^2 \neq 0$. Then R has a nonzero idempotent e, and by Albert's theorem [1] R/N is a field. Since R/N can be embedded in the real closure of $\mathbb{Q}$ [6, p.285] we may assume that it is a subfield of the reals.

Suppose first that R is unital and $N \neq 0$. If $F = R/N$ is a proper extension of $\mathbb{Q}$ let $\{a_i : i \geq 1\}$ be a basis of ${}_{\mathbb{Q}}F$ with $a_1 = 1$. Then $F = \underset{\rightarrow}{\oplus}_{\ i \geq 1}\ \mathbb{Q}a_i$ is a totally ordered group. Since $N^2 = 0 \quad N$ is a vector space over F. Let $0 \neq x \in N$. Then F^+x is a partial order of R and hence is contained in a total order T of R. This total order induces a total order T_F of the field F. Since (F, T_F) is archimedean, $T_F \not\subseteq F^+$. Let $a \in T$ with $a + N \notin F^+$. Then $(a+N)x = ax \in T \cap Fx = F^+x$ yields the contradiction that $a + N \in F^+$. Thus $F = \mathbb{Q}$ and $R = \mathbb{Q}1 + N$ is isomorphic to a ring of type (iii).

Suppose now that R is not unital. Since the left and right annihilator ideals of R are convex ideals one of them is contained in the other. Suppose that the right annihilator $r(R)$ is contained in the left annihilator $l(R)$. According to [2, Theorem 9.4.15] (also see [5, 2.4]) the Pierce decomposition of R is $R = B \oplus C \oplus D$ where $B = eRe,\ D = r(R) = (1-e)R,\ C = eR(1-e)\ $ and $\ C \oplus D = l(R) = R(1-e)$. Also, any total order of R is of the form $(B \underset{\rightarrow}{\oplus} C \underset{\rightarrow}{\oplus} D)^+$. If $C \neq 0\ $ and $\ D \neq 0$, then a total order $(C \underset{\leftarrow}{\oplus} D)^+$ of $C \oplus D$ could be extended to a total order of R. Thus one of C or D is zero but the other is nonzero. If $C = 0$ then B and D are ideals of R. If B is not a field and $0 \neq b \in B$ with $b^2 = 0$ and $0 \neq d \in D$, then the partial order $(\mathbb{Z}b \underset{\leftarrow}{\oplus} \mathbb{Z}d)^+$ could be extended to a total order of R. Thus B is a field and R is of type (i). Suppose then that $D = 0$. If $0 \neq b \in B$ with $b^2 = 0$, then, since in any total order of R either $b \geq C$ or $\ -b \geq C$, we must have $bC = 0$. But then for $0 \neq c \in C$ there is a total order of R containing $(\mathbb{Z}b \underset{\leftarrow}{\oplus} \mathbb{Z}c)^+$. So B is a field. By an argument similar to the one given when R is unital we see that $B = \mathbb{Q}$. Thus R is of type (ii).

We next show that each of these algebras is an O^*- algebra. If $R = F \oplus N$ is of type (i) and P is a partial order of R, then $P_F = \{\alpha \in F : \alpha + x \in P\ \text{ for some }\ x \in N\}$ is a partial order of F. For P_F is closed under addition and multiplication; and if $\alpha + x\ $ and $\ -\alpha + y\ $ are in $P\ $ then $\ -\alpha^2 \in P \cap F$. Thus $\alpha = 0$ since F is an O^*- field. Now, if T_F is a total order of the field F with $T_F \supseteq P_F\ $ and $\ T_N\ $ is a total order of the group N with $T_N \supseteq P \cap N,\ $ then $\ R^+ = [(F,\ T_F) \underset{\rightarrow}{\oplus} (N, T_N)]^+$ is a total order of R which contains P.

Suppose that R is of type (iii) and that P is a partial order of R. If $x = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in P$ with $a < 0$ then we may assume that $a = -1$. But then $-1 = x^2 + 2x \in P$ and this is impossible. So if T_N is a total order of the group $\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ which contains $\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \cap P$, then $\mathbb{Q}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{\oplus}{\to} (\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}, T_N)$ gives a total order of R which contains P. Similarly, each ring of type (ii) is an O^*- ring.

It is interesting to note that the unique totally ordered (right or left) self-injective rings that are not unital are O^*- rings [8, Theorem 5.4].

A well-known result of Serre's [4, p.117] implies that a real algebraic extension of $\mathbb{Q}$ is an O^*- field if and only if each of the algebraic number fields that it contains is an O^*- field.

Example. Each real quadratic extension of $\mathbb{Q}$ is an O^*- field.

To see this we may assume that $F = \mathbb{Q}(\sqrt{e})$ where $e \in \mathbb{Z}^+$ is square-free. Let P be a partial order of F. By replacing P by $\mathbb{Q}^+P + \mathbb{Q}^+$ we may assume that $\mathbb{Q}^+P \subseteq P$ and $1 \in P$. Now, F has exactly two total orders [6, p. 287]: $T_1 = F \cap \mathbb{R}^+$ and $T_2 = \{a + b\sqrt{e} : a - b\sqrt{e} \in \mathbb{R}^+, a, b \in \mathbb{Q}\}$ where $\mathbb{R}^+$ is the total order of $\mathbb{R}$. All of the inequalities that subsequently appear will refer to this total order. If $x = a + b\sqrt{e}$, then $\bar{x} = a - b\sqrt{e}$. We first note that

(∗) If $x = a + b\sqrt{e} \in P$ with $a < 0$, then $x\bar{x} = a^2 - b^2e < 0$ and :

(∗1) $b > 0 \Leftrightarrow \sqrt{e} \in P \Leftrightarrow \bar{x} < 0$,

(∗2) $b < 0 \Leftrightarrow -\sqrt{e} \in P \Leftrightarrow x < 0$.

For $b\sqrt{e} = x - a \in P$; so $b > 0$ (respectively, $b < 0$) $\Leftrightarrow \sqrt{e}$ (respectively, $-\sqrt{e}$) $\in P$. Also, $x^2 - ax = eb^2 + ab\sqrt{e} \in P$; so $1 + \frac{a}{eb}\sqrt{e}$, $-(1 + \frac{b}{a}\sqrt{e}$) $\in P$, and consequently $(\dfrac{a^2 - b^2e}{abe})\sqrt{e} = (\dfrac{a}{eb} - \dfrac{b}{a})\sqrt{e} \in P$. Thus, $a^2 - b^2e < 0$ in both cases. If $x < 0$ and also $b > 0$, then $b^2e < a^2$; so $b < 0$. Trivially, $b < 0$ gives $x < 0$. The other case is similar.

Suppose that $P \not\subseteq T_1, T_2$. Then there are $x \in P \setminus T_1$ and $y \in P \setminus T_2$. So $x = a + b\sqrt{e} < 0$ and $y = c + d\sqrt{e}$ with $\bar{y} < 0$; hence $a < 0$ or $b < 0$, and $c < 0$ or $d > 0$. We consider each of the four cases separately.

(I) $a < 0$ and $c < 0$. This case is impossible by (∗1) and (∗2).

(II) $a < 0$ and $d > 0$. By (∗2) $-\sqrt{e} \in P$ and hence $c > 0$. But then $y_1 = -\sqrt{e}y = -de - c\sqrt{e} \in P$ and $\bar{y}_1 = -de + c\sqrt{e} = \sqrt{e}\bar{y} < 0$. This is case I.

(III) $b < 0$ and $c < 0$. After passing to $\bar{P}$ this is case II.

(IV) $b < 0$ and $d > 0$. To avoid the other cases $a \geq 0$ and $c \geq 0$. If $a = 0$ then $-\sqrt{e} \in P$, and hence $-\sqrt{e}y = -de - c\sqrt{e} \in P$; so $c^2e > d^2e^2$ and $c^2 > d^2e$ by (∗). But $c < d\sqrt{e}$ since $\bar{y} < 0$. Thus $a > 0$. If $c = 0$, then $\sqrt{e} \in P$ and $\sqrt{e}x = be + a\sqrt{e} \in P$; and hence $y \in P$ gives case II. Thus $c > 0$ and $xy = (ac + bde) + (ad + bd)\sqrt{e} \in P$ with $ac + bde < 0$, since $a < -b\sqrt{e}$ and $c < d\sqrt{e}$. By (∗1) and (∗2) $\sqrt{e} \in P$ or $-\sqrt{e} \in P$. If the former holds then $\sqrt{e}x = be + a\sqrt{e} \in P$; this is case II. If the latter holds, $y_1 = -\sqrt{e}y = -de - c\sqrt{e} \in P$ and $\bar{y}_1 < 0$. This contradicts (∗1).

This calculation actually gives the

Corollary. *The following statements are equivalent for the quadratic extension* $F = K(\sqrt{e})$ *of the* O^**- field* K*.*

(1) F *is an* O^**- field.*

(2) e *is totally positive in* K *(that is,* e *is positive in each total order of* K*), and for each partial order* P *of* F *there is a total order* T *of* K *such that* PT *is a partial order of* F*.*

(3) *Each maximal partial order of* F *contains* e *and a total order of* K*.*

Note that $\mathbb{Q}(\sqrt[4]{e})$ is not an O^*- field if $0 < e \in \mathbb{Z}$ is square-free.

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