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A CHARACTERIZATION OF SEMIBOUNDED SELFADJOINT OPERATORS

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Abstract. For a class of closed symmetric operators S with defect numbers $(1,1)$ it is possible to define a generalization of the Friedrichs extension, which coincides with the usual Friedrichs extension when S is semibounded. In this paper we provide an operator-theoretic interpretation of this class of symmetric operators. Moreover, we prove that a selfadjoint operator A is semibounded if and only if each one-dimensional restriction of A has a generalized Friedrichs extension.

0. Introduction

Let A be a selfadjoint relation in a Hilbert space $\mathfrak{H}$ with inner product $[.,.]$. The one-dimensional symmetric restrictions of A are in one-to-one correspondence with all one-dimensional subspaces $\operatorname{span}\{\varphi\}\subset\mathfrak{H}$, via

$$S=\{\{f,g\}\in A:\ [g-\bar{\mu}f,\varphi]=0\}, \tag{0.1}$$

where $\mu\in\mathbb{C}\setminus\mathbb{R}$ is fixed. If A is an operator and $E(t)$ is its spectral function, then

$$f\in\operatorname{dom}S \text{ if and only if } f\in\operatorname{dom}A \text{ and } \textstyle\int_{\mathbb{R}}(t-\bar{\mu})\,d([E(t)f,\varphi])=0. \tag{0.2}$$

If A is a semibounded selfadjoint relation, then each symmetric restriction S is semibounded and

$$S_F=\{\{f,g\}\in S^*:\ f\in\mathfrak{H}_S\} \tag{0.3}$$

is a selfadjoint extension of S, called the Friedrichs extension. Here the Hilbert space $\mathfrak{H}_S$ is the completion of $\operatorname{dom}S$ with the inner product $[(S+a)f,g]$, $f,g\in\operatorname{dom}S$, where $a\in\mathbb{R}$ is any number such that $S+a$ has a positive lower bound. The extension S_F is the unique selfadjoint extension H of S with the property $\operatorname{dom}H\subset\mathfrak{H}_S$, see e.g. [1] and [6].

We now introduce a generalization of the Friedrichs extension for symmetric operators, which are not necessarily semibounded. Let S be a closed symmetric

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operator with defect numbers $(1,1)$ and let A be a selfadjoint extension of S. Let $\mathfrak{H}_{+1,A}$ be the Hilbert space $\operatorname{dom}|A_s|^{\frac{1}{2}}$ provided with the inner product

$$[f,g] + [|A_s|^{\frac{1}{2}}f, |A_s|^{\frac{1}{2}}g], \quad f,g \in \operatorname{dom}|A_s|^{\frac{1}{2}}, \tag{0.4}$$

where A_s denotes the operator part of A. Let $\mathfrak{H}_{S,A}$ be the closure of $\operatorname{dom} S$ in $\mathfrak{H}_{+1,A}$. Then

$$A_S = \{\{f,g\} \in S^* : \, f \in \mathfrak{H}_{S,A}\} \tag{0.5}$$

defines a selfadjoint extension of S. In fact, A_S is the only selfadjoint extension H of S with the property that $\operatorname{dom} H \subset \mathfrak{H}_{S,A}$. The extension A_S of S may or may not depend on A. More precisely, we show that the following alternative holds: either $A_S = A$ for each selfadjoint extension A of S or A_S does not depend on the selfadjoint extension A. In this last case A_S is called the generalized Friedrichs extension of S, cf. also [7]. We then prove that a selfadjoint operator A is semibounded if and only if each one-dimensional symmetric restriction of A has a generalized Friedrichs extension.

We indicate the connection with the function-theoretic approach to the above alternative. If S in (0.1) is determined by $\varphi \in \mathfrak{H}$, then the corresponding nullspace $\ker(S^* - \ell)$ is spanned by

$$\chi(\ell) = (I + (\ell - \mu)(A - \ell)^{-1})\varphi, \quad \ell \in \mathbb{C} \setminus \mathbb{R},$$

so that $\varphi = \chi(\mu)$. For $a \in \mathbb{R}$ the function

$$Q(\ell) = a + \bar{\mu}[\varphi,\varphi] + (\ell - \bar{\mu})[(I + (\ell - \mu)(A - \ell)^{-1})\varphi, \varphi] \tag{0.6}$$

is a Q-function of A and S; by definition it is a solution of

$$\frac{Q(\ell) - \overline{Q(\lambda)}}{\ell - \bar{\lambda}} = [\chi(\ell), \chi(\lambda)].$$

It belongs to the class $\mathbf{N}$ of Nevanlinna functions, i.e., the class of functions $Q(\ell)$, defined and holomorphic on $\mathbb{C} \setminus \mathbb{R}$, for which $\overline{Q(\ell)} = Q(\bar{\ell})$ and $\operatorname{Im} Q(\ell)/\operatorname{Im} \ell \geq 0$, $\ell \in \mathbb{C} \setminus \mathbb{R}$. We will say that S is of category $\mathbf{N}_1$ if there is a selfadjoint extension A whose Q-function belongs to the class $\mathbf{N}_1$, i.e., if it satisfies $\int_1^\infty \operatorname{Im} Q(iy)/y \, dy < \infty$, which is equivalent to $\varphi \in \operatorname{dom}|A|^{\frac{1}{2}}$. Hence, S is of category $\mathbf{N}_1$ if and only if for some Q-function $Q(\ell)$ the function $(Q(\ell) - \tau(\operatorname{Im} Q(\mu))^2)/(\tau Q(\ell) + 1)$ belongs to $\mathbf{N}_1$ for some $\tau \in \mathbb{R} \cup \{\infty\}$, see [3]. Moreover, S is of category $\mathbf{N}_0$ if there is a selfadjoint extension A whose Q-function belongs to the class $\mathbf{N}_0$, i.e., if it satisfies $\sup_{y>0} y \operatorname{Im} Q(iy) < \infty$, which is equivalent to $\varphi \in \operatorname{dom} A$. If $Q(\ell) \in \mathbf{N}_1$, then $\lim_{y\to\infty} Q(iy)$ exists and is real, see [5]. The above alternative for a closed symmetric operator S may now be stated as follows: either S is not of category $\mathbf{N}_1$, in which case all Q-functions belong to $\mathbf{N} \setminus \mathbf{N}_1$, or S is of category $\mathbf{N}_1$, in which case all but one of its selfadjoint extensions A have a Q-function belonging to $\mathbf{N}_1$. In this last case, the exceptional selfadjoint extension is the generalized Friedrichs extension; its Q-function has the property $\lim_{y\to\infty} Q(iy) = \infty$. For this function-theoretic approach to generalized Friedrichs extensions we refer to [2], [3].

In Section 1 we prove the above alternative for a symmetric operator in operator-theoretic terms. In Section 2 we prove some auxiliary results concerning a special class of finite measures, which may be of independent interest. The characterization of semibounded selfadjoint operators is given in Section 3.

1. Generalized Friedrichs extensions

Let S be a closed symmetric operator with defect numbers $(1,1)$ in a Hilbert space $\mathfrak{H}$. In this section we explain the alternative concerning generalized Friedrichs extensions in operator-theoretic terms. Note that S need not be densely defined, so that it may have a proper selfadjoint relation extension A in $\mathfrak{H}$. This means that the multivalued part $\operatorname{mul} A = \{g \in \mathfrak{H} : \{0,g\} \in A\}$ is nontrivial; it reduces the relation A and the restriction of A to $\mathfrak{H} \ominus \operatorname{mul} A$ is the operator part, denoted by A_s, so that $\operatorname{dom} A_s = \operatorname{dom} A$ is dense in $\mathfrak{H} \ominus \operatorname{mul} A$.

First assume that A is a selfadjoint operator extension of S. Let $\mathfrak{H}_{+1,A}$ be the Hilbert space as defined in (0.4). Clearly, $\operatorname{dom} A$ is dense in the Hilbert space $\mathfrak{H}_{+1,A}$. If $\chi(\mu) \in \mathfrak{H}_{+1,A}$ for some $\mu \in \mathbb{C} \setminus \mathbb{R}$, then the identity

$$\frac{\chi(\ell) - \chi(\mu)}{\ell - \mu} = (A - \ell)^{-1}\chi(\mu) \tag{1.1}$$

shows that $\chi(\ell) \in \mathfrak{H}_{+1,A}$ for all $\ell \in \mathbb{C} \setminus \mathbb{R}$. Since S has defect numbers $(1,1)$ the codimension of $\mathfrak{H}_{S,A}$ in $\mathfrak{H}_{+1,A}$ is at most one. In fact, if $\chi(\ell) \notin \mathfrak{H}_{+1,A}$, $\ell \in \mathbb{C} \setminus \mathbb{R}$, then

$$\mathfrak{H}_{S,A} = \mathfrak{H}_{+1,A}, \tag{1.2}$$

while if $\chi(\ell) \in \mathfrak{H}_{+1,A}$, $\ell \in \mathbb{C} \setminus \mathbb{R}$, then

$$\mathfrak{H}_{S,A} \subset \mathfrak{H}_{+1,A}, \quad \mathfrak{H}_{S,A} \neq \mathfrak{H}_{+1,A}, \tag{1.3}$$

see [2, Theorem 3.1]. It follows from (1.2) and (1.3) that

$$\ker(S^* - \ell) \cap \mathfrak{H}_{S,A} = \{0\}, \quad \ell \in \mathbb{C} \setminus \mathbb{R}. \tag{1.4}$$

To see this, suppose that $\chi(\mu) \in \mathfrak{H}_{S,A}$ for some $\mu \in \mathbb{C} \setminus \mathbb{R}$. This is impossible when $\chi(\ell) \notin \mathfrak{H}_{+1,A}$, $\ell \in \mathbb{C} \setminus \mathbb{R}$. In the case that $\chi(\ell) \in \mathfrak{H}_{+1,A}$, $\ell \in \mathbb{C} \setminus \mathbb{R}$, it follows by continuity from (0.2) that

$$h \in \mathfrak{H}_{S,A} \text{ if and only if } h \in \mathfrak{H}_{+1,A} \text{ and } \int_{\mathbb{R}} (t - \bar{\mu})\, d([E(t)h, \chi(\mu)]) = 0. \tag{1.5}$$

Now $\chi(\mu) \in \mathfrak{H}_{S,A}$ would imply

$$\int_{\mathbb{R}} (t - \bar{\mu})\, d([E(t)\chi(\mu), \chi(\mu)]) = 0,$$

a contradiction.

Now assume that A is a selfadjoint extension of S, which is a proper relation. In this case S is nondensely defined and S^* is the relation (multivalued operator) defined by

$$S^* = \{\{f,g\} \in \mathfrak{H}^2 : [g,h] = [f,k], \text{ for all } \{h,k\} \in S\}.$$

Since S has defect numbers $(1,1)$, $\operatorname{mul} S^* = \{g \in \mathfrak{H} : \{0,g\} \in S^*\}$ is one-dimensional and A is the only proper relation extension of S; it is given by

$$S \dot{+} (\{0\} \oplus \operatorname{mul} S^*), \tag{1.6}$$

where $\dot{+}$ stands for the componentwise sum in $\mathfrak{H}^2$. Hence $\operatorname{mul} A = \operatorname{mul} S^*$ and $A_s = PS$, where P is the orthogonal projection onto $\mathfrak{H} \ominus \operatorname{mul} S^*$. Observe that (1.1) still makes sense with $\chi(\ell) \in \ker(S^* - \ell)$, $\ell \in \mathbb{C} \setminus \mathbb{R}$, and that $\chi(\ell) \notin \overline{\operatorname{dom}}\, A$, cf. [4]. Since $\mathfrak{H}_{1,A} \subset \mathfrak{H} \ominus \operatorname{mul} A = \overline{\operatorname{dom}}\, A$, we have $\chi(\ell) \notin \mathfrak{H}_{1,A}$ and (1.4) is still valid. Moreover, $\operatorname{dom} S = \operatorname{dom} A = \operatorname{dom} A_s$ shows that (1.2) holds.

For any selfadjoint extension A of S, let A_S be the relation associated with S and A by (0.5). Then clearly, $S \subset A_S \subset S^*$. In order to show that A_S is selfadjoint we use von Neumann's formula. There is a one-to-one correspondence between all selfadjoint extensions of S and all $\zeta \in \mathbb{C}$ with $|\zeta| = 1$ via

$$A(\zeta) = S \dot{+} \operatorname{span}\{\chi(\bar{\mu}) - \zeta\chi(\mu), \bar{\mu}\chi(\bar{\mu}) - \zeta\mu\chi(\mu)\}. \tag{1.7}$$

Here $\mu \in \mathbb{C} \setminus \mathbb{R}$ and $\chi(\mu)$ and $\chi(\bar{\mu})$ have the same norm, since $I + (\bar{\mu} - \mu)(A - \bar{\mu})^{-1}$ is a unitary operator. It follows from (1.1) that $\{\chi(\bar{\mu}) - \chi(\mu), \bar{\mu}\chi(\bar{\mu}) - \mu\chi(\mu)\} \in A$, so that A itself corresponds to $\zeta = 1$.

Theorem 1.1. *The relation A_S is a selfadjoint extension of S. In fact, it is the only selfadjoint extension H of S with the property that* $\operatorname{dom} H \subset \mathfrak{H}_{S,A}$. *The identity*

$$\mathfrak{H}_{S,A} = \mathfrak{H}_{+1,A} \tag{1.8}$$

holds if and only if $A_S = A$.

Proof. It follows from (1.4) that there is at most one $\zeta \in \mathbb{C}$ for which

$$\chi(\bar{\mu}) - \zeta\chi(\mu) \in \mathfrak{H}_{S,A}. \tag{1.9}$$

We claim that there is precisely one $\zeta \in \mathbb{C}$ for which (1.9) holds and we show in that case that $|\zeta| = 1$. It then follows from von Neumann's formula that A_S is selfadjoint. Moreover, if H is a selfadjoint extension of S such that $\operatorname{dom} H \subset \mathfrak{H}_{S,A}$, then $H \subset S^*$ implies that $H \subset A_S$. Since both H and A_S are selfadjoint, we obtain $H = A_S$.

To prove our claim, we first consider the case that (1.8) holds. Then, clearly,

$$A_S = \{\{f, g\} \in S^* : f \in \mathfrak{H}_{+1,A}\}.$$

Since $\chi(\bar{\mu}) - \chi(\mu) \in \operatorname{dom} A \subset \mathfrak{H}_{+1,A} = \mathfrak{H}_{S,A}$, it follows that $\zeta = 1$ satisfies (1.9).

Next assume that $\mathfrak{H}_{S,A}$ is a proper subspace of $\mathfrak{H}_{+1,A}$, or equivalently that $\chi(\ell) \in \mathfrak{H}_{+1,A}$, $\ell \in \mathbb{C} \setminus \mathbb{R}$, cf. (1.2) and (1.3). Then the integrals in the following equation are well-defined:

$$\int_{\mathbb{R}} (t - \mu)\, d([E(t)\chi(\mu), \chi(\mu)]) = \zeta \int_{\mathbb{R}} (t - \bar{\mu})\, d([E(t)\chi(\mu), \chi(\mu)]). \tag{1.10}$$

This equation defines a unique $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. However, using (1.1) with $\ell = \bar{\mu}$ we see that (1.10) is equivalent to

$$\int_{\mathbb{R}} (t - \bar{\mu})\, d([E(t)(\chi(\bar{\mu}) - \zeta\chi(\mu)), \chi(\mu)]) = 0.$$

Due to (1.5) we conclude that $\chi(\bar{\mu}) - \zeta\chi(\mu) \in \mathfrak{H}_{S,A}$, so that also in this case (1.9) is satisfied for $|\zeta| = 1$.

Finally, note that in the first case when (1.8) holds we have $\zeta = 1$ so that $A_S = A$, while in the opposite case the identity (1.10) shows that $\zeta \neq 1$ and therefore $A_S \neq A$. □

Theorem 1.2. *Let S be a symmetric operator with defect numbers* $(1, 1)$. *Then the following alternative holds:*

(i) *The identity* (1.8) *holds for all selfadjoint extensions A of S and for each selfadjoint extension A of S we have $A_S = A$.*
(ii) *The identity* (1.8) *holds for precisely one selfadjoint extension of S. For each selfadjoint extension A of S the relation A_S is independent of A.*

Proof. If (1.8) holds for all selfadjoint extensions A of S, then according to Theorem 1.1, $A_S = A$ for each selfadjoint extension A of S.

Now let A be a selfadjoint extension of S for which (1.8) is not valid (so that A and A_S are different). Then $\chi(\ell) \in \mathfrak{H}_{+1,A}$, $\ell \in \mathbb{C} \setminus \mathbb{R}$, and it follows from [2, Theorem 4.1], that

$$\operatorname{dom} |A(\tau)|^{\frac{1}{2}} = \operatorname{dom} |A|^{\frac{1}{2}},$$

for all $\tau \in \mathbb{R} \cup \{\infty\}$, $\tau \neq \tau_0$, where $\tau_0 \in \mathbb{R} \cup \{\infty\}$ is the exceptional parameter. By the closed graph theorem, the (topological) spaces $\mathfrak{H}_{+1,A(\tau)}$ and hence, in particular, the spaces $\mathfrak{H}_{S,A(\tau)}$ as point sets are the same for all $\tau \neq \tau_0$. Therefore, $A(\tau)_S = A_S$ for all $\tau \neq \tau_0$. Since all $A(\tau)$, $\tau \neq \tau_0$, are different from A_S and since A_S is a selfadjoint extension of S, it follows that A_S is the exceptional extension corresponding to τ_0. For the exceptional selfadjoint extension $A(\tau_0)$ we have $\chi(\ell) \notin \mathfrak{H}_{+1,A(\tau_0)}$, see [3] for the case that $A(\tau_0)$ is an operator and see the above discussion for the case that $A(\tau_0)$ is a proper relation. Hence, equality (1.8) holds and by Theorem 1.1, $A(\tau_0)_S = A(\tau_0) = A_S$.

Thus, the case where the alternative (i) does not hold gives rise to the alternative (ii). □

Due to (1.2) and (1.3) the alternative in Theorem 1.2 can be restated as:

(i) For all selfadjoint extensions of S the corresponding Q-functions belong to $\mathbf{N} \setminus \mathbf{N}_1$.

(ii) There is precisely one selfadjoint extension (the generalized Friedrichs extension) of S for which the corresponding Q-function belongs to $\mathbf{N} \setminus \mathbf{N}_1$.

If S is semibounded, the space $\mathfrak{H}_{S,A}$ is the completion of $\operatorname{dom} S$ with respect to the inner product $[(S+a)f, g]$, $f, g \in \operatorname{dom} S$, with a large enough; hence, it is independent of any selfadjoint extension of S. In particular, S satisfies the second alternative of Theorem 1.2 and the generalized Friedrichs extension A_S coincides with S_F in (0.3). If S is nondensely defined (but not necessarily semibounded), it also satisfies the second alternative of Theorem 1.2 and the generalized Friedrichs extension A_S is given by (1.6). In this case, all but one of the selfadjoint extensions of S are operators and have a Q-function in $\mathbf{N}_0 \subset \mathbf{N}_1$. If, in addition, S is also semibounded, then (1.6) is the Friedrichs extension S_F of S, cf. [1].

2. Some auxiliary results

If $d\mu$ is a finite measure and if h is a nonnegative, bounded measurable function on $\mathbb{R}$, then $hd\mu$ is a finite measure and, hence, the function

$$Q^{hd\mu}(\ell) = \int_{\mathbb{R}} \left(\frac{1}{t-\ell} - \frac{t}{t^2+1} \right) (t^2+1) h(t) d\mu(t), \quad \ell \in \mathbb{C} \setminus \mathbb{R}, \tag{2.1}$$

is well-defined and belongs to $\mathbf{N}$. Moreover, it belongs to $\mathbf{N} \setminus \mathbf{N}_1$ if and only if the function h satisfies

$$\int_{\mathbb{R}} |t| h(t)\, d\mu(t) = \infty, \tag{2.2}$$

which is certainly true when

$$\int_0^\infty |t| h(t)\, d\mu(t) = \infty, \quad \int_{-\infty}^0 |t| h(t)\, d\mu(t) = \infty. \tag{2.3}$$

In this section we consider finite measures $d\mu$ which satisfy the additional conditions:

$$\int_0^\infty |t|\, d\mu(t) = \infty, \quad \int_{-\infty}^0 |t|\, d\mu(t) = \infty. \tag{2.4}$$

These conditions guarantee the existence of a nonnegative, bounded measurable function h on $\mathbb{R}$ for which the corresponding function $Q^{hd\mu}(\ell)$ in (2.1) has some special limiting properties.

Lemma 2.1. *Let $d\mu$ be a finite measure which satisfies* (2.4). *Then there exists a measurable function h on $\mathbb{R}$ with $0 \leq h(t) \leq 1$, $t \in \mathbb{R}$, such that* (2.3) *holds and such that* $\lim_{y\to\infty} \operatorname{Im} Q^{hd\mu}(iy) = 0$.

Proof. Define the nonnegative function $\iota(t, y)$ by

$$\iota(t, y) = y\, \frac{t^2 + 1}{t^2 + y^2}, \quad y > 0,\ t \in \mathbb{R}.$$

Clearly, we have, for $y \geq 1$,

$$\iota(t, y) \leq y, \quad t \in \mathbb{R}, \tag{2.5}$$

and

$$\iota(t, y) \leq t^2 + 1, \quad \lim_{y\to\infty} \iota(t, y) = 0, \quad t \in \mathbb{R}. \tag{2.6}$$

Note that for all $y \in \mathbb{R}$,

$$\iota(t, y) \leq |t|, \quad \text{for} \quad |t| \geq 1. \tag{2.7}$$

We construct the function h inductively. Let $u_1 = 1$. Due to (2.4) there exists $v_1 > u_1$ such that

$$m_1^- = \int_{[-v_1,-u_1]} |t| d\mu(t) \geq 1, \quad m_1^+ = \int_{[u_1,v_1]} |t| d\mu(t) \geq 1. \tag{2.8}$$

Define the function h_1 by

$$h_1(t) = \frac{1}{m_1^-} 1_{[-v_1,-u_1]} + \frac{1}{m_1^+} 1_{[u_1,v_1]}, \tag{2.9}$$

so that h_1 has its support in $[-v_1, -u_1] \cup [u_1, v_1]$ and $|h_1(t)| \leq 1$, $t \in \mathbb{R}$.

Assume that the function h_{n-1} has been defined, with support contained in

$$\sum_{k=1}^{n-1} ([-v_k, -u_k] \cup [u_k, v_k]) \tag{2.10}$$

(disjoint union). Since the measure $d\mu$ is finite, we can choose $u_n > v_{n-1}$ so large that

$$\left(\int_{(-\infty,-u_n]} + \int_{[u_n,\infty)} \right) \iota(t, y)\, d\mu(t) \leq \frac{1}{n}, \quad \text{for all } 1 \leq y \leq u_{n-1}, \tag{2.11}$$

due to (2.5), and such that

$$\int_{[-v_{n-1},v_{n-1}]} \iota(t, y)\, d\mu(t) \leq \frac{1}{n}, \quad \text{for all} \quad y \geq u_n, \tag{2.12}$$

due to (2.6) and dominated convergence. Moreover, there exists $v_n > u_n$ such that

$$m_n^- = \int_{[-v_n,-u_n]} |t| d\mu(t) \geq 1, \quad m_n^+ = \int_{[u_n,v_n]} |t| d\mu(t) \geq 1, \tag{2.13}$$

due to (2.4). Define

$$h_n = h_{n-1} + \frac{1}{nm_n^-} 1_{[-v_n,-u_n]} + \frac{1}{nm_n^+} 1_{[u_n,v_n]}, \tag{2.14}$$

so that the supports of the summands are disjoint. Clearly (u_n) is a monotonically increasing sequence, with $u_n \to \infty$ as $n \to \infty$, on account of (2.13). The inductive argument shows that $h(t) = \lim_{n\to\infty} h_n(t)$ is measurable and $0 \leq h(t) \leq 1$, $t \in \mathbb{R}$. It follows from (2.13) and (2.14) that

$$\int_0^\infty |t| h(t)\, d\mu(t) = \int_{-\infty}^0 |t| h(t)\, d\mu(t) = \sum_{k=1}^\infty \frac{1}{k} = \infty,$$

so that (2.3) is satisfied.

Finally, we prove the announced limiting behaviour of $Q^{hd\mu}(\ell)$. Let $y \geq 1$ and assume that $y \in [u_n, u_{n+1}]$ (with $n > 1$) and write

$$\begin{aligned}
\operatorname{Im} Q^{hd\mu}(iy) &= \int_{\mathbb{R}} \iota(t,y) h(t)\, d\mu(t) \\
&= \int_{[-v_{n-1},v_{n-1}]} \iota(t,y) h(t)\, d\mu(t) \\
&\quad + \left(\int_{[-v_{n+1},-u_n]} + \int_{[u_n,v_{n+1}]} \right) \iota(t,y) h(t)\, d\mu(t) \\
&\quad + \left(\int_{(-\infty,-u_{n+2}]} + \int_{[u_{n+2},\infty)} \right) \iota(t,y) h(t)\, d\mu(t).
\end{aligned}$$

Then, due to (2.12) we have

$$\int_{[-v_{n-1},v_{n-1}]} \iota(t,y) h(t)\, d\mu(t) \leq \int_{[-v_{n-1},v_{n-1}]} \iota(t,y)\, d\mu(t) \leq \frac{1}{n}.$$

It follows from (2.7), that

$$\begin{aligned}
&\left(\int_{[-v_{n+1},-u_n]} + \int_{[u_n,v_{n+1}]} \right) \iota(t,y) h(t)\, d\mu(t) \\
&\qquad \leq \left(\int_{[-v_{n+1},-u_n]} + \int_{[u_n,v_{n+1}]} \right) |t| h(t)\, d\mu(t) = \frac{2}{n} + \frac{2}{n+1},
\end{aligned}$$

where the last equality follows from (2.13) and (2.14). Finally, due to (2.11) we have

$$\begin{aligned}
&\left(\int_{(-\infty,-u_{n+2}]} + \int_{[u_{n+2},\infty)} \right) \iota(t,y) h(t)\, d\mu(t) \\
&\qquad \leq \left(\int_{(-\infty,-u_{n+2}]} + \int_{[u_{n+2},\infty)} \right) \iota(t,y)\, d\mu(t) \leq \frac{1}{n+2}.
\end{aligned}$$

Hence for the values $y \in [u_n, u_{n+1}]$ we obtain an estimate

$$\operatorname{Im} Q^{hd\mu}(iy) \leq \frac{3}{n} + \frac{2}{n+1} + \frac{1}{n+2}.$$

Letting $n \to \infty$ so that $y \to \infty$ we conclude that $\lim_{y\to\infty} \operatorname{Im} Q^{hd\mu}(iy) = 0$. □

The above lemma gives a construction of a Nevanlinna function whose imaginary part has a specific limiting behaviour. Similarly, we can construct a Nevanlinna function whose real part has a specific limiting behaviour. However, the arguments have to be suitably modified as the reasoning is a little more subtle.

Lemma 2.2. *Let $d\mu$ be a finite measure which satisfies* (2.4)*. Then there exists a measurable function h on $\mathbb{R}$ with $0 \leq h(t) \leq 1$, $t \in \mathbb{R}$, such that* (2.3) *holds and such that $\lim_{y\to\infty} \operatorname{Re} Q^{hd\mu}(iy) = 0$.*

Proof. Define the function $\rho(t, y)$ by

$$\rho(t, y) = |t| \, \frac{(y^2 - 1)}{t^2 + y^2}, \quad y > 0, \quad t \in \mathbb{R}.$$

When $y \geq 1$ this function is nonnegative and, moreover,

$$\rho(t, y) \leq \frac{y^2 - 1}{2y}, \quad t \in \mathbb{R} \tag{2.15}$$

(where the upper bound itself is monotonically increasing with y), and

$$|t| - \rho(t, y) = |t| \frac{t^2 + 1}{t^2 + y^2} \leq |t|, \quad \lim_{y\to\infty} (|t| - \rho(t, y)) = 0, \quad t \in \mathbb{R}. \tag{2.16}$$

Note that

$$\rho(t, y) \leq |t|, \quad y \geq 1. \tag{2.17}$$

Now we construct the function h. Let $u_1 = 1$. Due to (2.4) there exists $v_1 > u_1$ such that (2.8) holds. Define the function h_1 by (2.9), so that h_1 has its support in $[-v_1, -u_1] \cup [u_1, v_1]$ and $|h_1(t)| \leq 1$, $t \in \mathbb{R}$.

Assume that the function h_{n-1} has been defined, with support contained in the set given by the disjoint union (2.10). Since the measure $d\mu$ is finite, we can choose $u_n > v_{n-1}$ so large that

$$\left(\int_{(-\infty, -u_n]} + \int_{[u_n, \infty)} \right) \rho(t, y) \, d\mu(t) \leq \frac{1}{n}, \quad \text{for all} \quad 1 \leq y \leq u_{n-1}, \tag{2.18}$$

due to (2.15), and such that

$$\int_{[-v_{n-1}, v_{n-1}]} (|t| - \rho(t, y)) \, d\mu(t) \leq \frac{1}{n}, \quad \text{for all} \quad y \geq u_n, \tag{2.19}$$

by (2.16) and dominated convergence. Moreover, due to (2.4), there exists $v_n > u_n$, such that (2.13) holds. Define the function h_n by (2.14) (so that the supports of the summands are disjoint). Clearly, (u_n) is a monotonically increasing sequence and $u_n \to \infty$ as $n \to \infty$, cf. (2.13). Moreover, $h(t) = \lim_{n\to\infty} h_n(t)$ is measurable and $0 \leq h(t) \leq 1$, $t \in \mathbb{R}$. It follows from (2.13) and (2.14) that for all $n \in \mathbb{N}$:

$$\int_{[0, v_n]} |t| h(t) \, d\mu(t) = \sum_{k=1}^{n} \frac{1}{k} = \int_{[-v_n, 0]} |t| h(t) \, d\mu(t). \tag{2.20}$$

In particular,

$$\int_{[0,\infty)} |t| h(t)\, d\mu(t) = \int_{(-\infty,0]} |t| h(t)\, d\mu(t) = \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

so that (2.3) is satisfied.

Finally, we prove the announced limiting behaviour of $Q^{hd\mu}(\ell)$. Let $y \geq 1$ and assume that $y \in [u_n, u_{n+1}]$. Then,

$$\begin{aligned}
\operatorname{Re} Q^{hd\mu}(iy) &= \int_{\mathbb{R}} t \frac{1-y^2}{(t^2+y^2)} h(t)\, d\mu(t) \\
&= \int_{[0,v_{n-1}]} (|t| - \rho(t,y)) h(t)\, d\mu(t) - \int_{[-v_{n-1},0]} (|t| - \rho(t,y)) h(t)\, d\mu(t) \\
&\quad - \int_{[0,v_{n-1}]} |t| h(t)\, d\mu(t) + \int_{[-v_{n-1},0]} |t| h(t)\, d\mu(t) \\
&\quad - \int_{[u_n,\infty)} \rho(t,y) h(t)\, d\mu(t) + \int_{(-\infty,-u_n]} \rho(t,y) h(t)\, d\mu(t).
\end{aligned}$$

Since the integrands of the summands are nonnegative we obtain by using (2.20)

$$\begin{aligned}
|\operatorname{Re} Q^{hd\mu}(iy)| \leq & \left(\int_{[-v_{n-1},0]} + \int_{[0,v_{n-1}]} \right) (|t| - \rho(t,y)) h(t)\, d\mu(t) \\
& + \left(\int_{[-v_{n+1},-u_n]} + \int_{[u_n,v_{n+1}]} \right) \rho(t,y) h(t)\, d\mu(t) \\
& + \left(\int_{(-\infty,-u_{n+2}]} + \int_{[u_{n+2},\infty)} \right) \rho(t,y) h(t)\, d\mu(t).
\end{aligned}$$

Due to (2.19),

$$\begin{aligned}
& \left(\int_{[-v_{n-1},0]} + \int_{[0,v_{n-1}]} \right) (|t| - \rho(t,y)) h(t) d\mu(t) \\
& \qquad \leq \int_{[-v_{n-1},v_{n-1}]} (|t| - \rho(t,y)) d\mu(t) \leq \frac{1}{n}.
\end{aligned}$$

Due to (2.17),

$$\begin{aligned}
& \left(\int_{[-v_{n+1},-u_n]} + \int_{[u_n,v_{n+1}]} \right) \rho(t,y) h(t)\, d\mu(t) \\
& \qquad \leq \left(\int_{[-v_{n+1},-u_n]} + \int_{[u_n,v_{n+1}]} \right) |t| h(t)\, d\mu(t) = \frac{2}{n} + \frac{2}{n+1},
\end{aligned}$$

where the last equality follows from (2.13) and (2.14). Due to (2.18),

$$\begin{aligned}
& \left(\int_{(-\infty,-u_{n+2}]} + \int_{[u_{n+2},\infty)} \right) \rho(t,y) h(t)\, d\mu(t) \\
& \qquad \leq \left(\int_{(-\infty,-u_{n+2}]} + \int_{[u_{n+2},\infty)} \right) \rho(t,y)\, d\mu(t) \leq \frac{1}{n+2}.
\end{aligned}$$

Hence we obtain

$$|\mathrm{Re}\, Q^{hd\mu}(iy)| \leq \frac{3}{n} + \frac{2}{n+1} + \frac{1}{n+2}, \quad \text{for } y \in [u_n, u_{n+1}].$$

Letting $n \to \infty$ so that $y \to \infty$ we conclude that $\lim_{y\to\infty} \mathrm{Re}\, Q^{hd\mu}(iy) = 0$. □

Proposition 2.3. *Let $d\mu$ be a finite measure which satisfies* (2.4). *Then there exists a measurable function with $0 \leq h(t) \leq 1$, $t \in \mathbb{R}$, such that* (2.3) *holds and such that $\lim_{y\to\infty} Q^{hd\mu}(iy) = 0$.*

Proof. Since $d\mu$ satisfies the assumptions of Lemma 2.1, there exists a measurable function h_1 on $\mathbb{R}$ with $0 \leq h_1(t) \leq 1$, $t \in \mathbb{R}$, such that

$$\lim_{y\to\infty} \mathrm{Im}\, Q^{h_1 d\mu}(iy) = 0, \quad \int_0^\infty |t| h_1(t)\, d\mu(t) = \infty, \quad \int_{-\infty}^0 |t| h_1(t)\, d\mu(t) = \infty.$$

The measure $h_1 d\mu$ satisfies the assumptions of Lemma 2.2 and hence there exists a measurable function h_2 on $\mathbb{R}$ with $0 \leq h_2(t) \leq 1$, $t \in \mathbb{R}$, such that

$$\lim_{y\to\infty} \mathrm{Re}\, Q^{h_2 h_1 d\mu}(iy) = 0, \int_0^\infty |t| h_2(t) h_1(t) d\mu(t) = \infty, \int_{-\infty}^0 |t| h_2(t) h_1(t) d\mu(t) = \infty.$$

Since $y\frac{t^2+1}{t^2+y^2} > 0$ for $y > 0$ and since $h_1(t)h_2(t) \leq h_1(t)$ we see that

$$\mathrm{Im}\, Q^{h_2 h_1 d\mu}(iy) \leq \mathrm{Im}\, Q^{h_1 d\mu}(iy),$$

for all $y > 0$. We conclude that $Q^{h_2 h_1 d\mu}(iy) \to 0$ as $y \to \infty$. □

3. A characterization of semiboundedness

In this section we characterize those selfadjoint operators which are semibounded or bounded in terms of symmetric one-dimensional restrictions.

We first state a result concerning selfadjoint operators which are not semibounded.

Proposition 3.1. *Let A be a selfadjoint operator and assume that A is not semibounded. Then there exists an element $\varphi \in \mathfrak{H} \setminus \mathrm{dom}\, |A|^{\frac{1}{2}}$, such that for the Q-function $Q(\ell)$ in* (0.6) *of S in* (0.1) *there is a finite limit $\lim_{y\to\infty} Q(iy) \in \mathbb{R}$.*

Proof. Let $(E_t)_{t\in\mathbb{R}}$ be the spectral family of A. Since A is not semibounded, the operators $AE_{(-\infty,0)}$ and $AE_{[0,\infty)}$ are each unbounded, and therefore $(-AE_{(-\infty,0)})^{\frac{1}{2}}$ and $(AE_{[0,\infty)})^{\frac{1}{2}}$ are unbounded. Hence there exists an element $\chi \in \mathfrak{H}$ such that

$$\int_0^\infty |t| d[E_t \chi, \chi] = \infty, \quad \int_{-\infty}^0 |t| d[E_t \chi, \chi] = \infty. \tag{3.1}$$

The Hilbert space $\mathfrak{K}$, defined by

$$\mathfrak{K} = \overline{\mathrm{span}}\, \{(A-\mu)(A-\ell)^{-1}\chi : \ell \in \mathbb{C} \setminus \mathbb{R}\},$$

reduces the selfadjoint operator A. Since $\chi \in \mathfrak{K}$, it follows that the restriction of A to $\mathfrak{K}$ is not semibounded. Therefore it suffices to prove Proposition 3.1 for the restriction of A to $\mathfrak{K}$. We define the measure $d\sigma$ by

$$d\sigma(t) = (t^2+1) d[E_t \chi, \chi],$$

so that

$$\int_{\mathbb{R}} \frac{d\sigma(t)}{t^2+1} < \infty.$$

We associate with $d\sigma$ the Hilbert space $L^2(d\sigma)$. Clearly, there is a unitary operator so that the triple

$$(\mathfrak{K}, A\,|_{\mathfrak{K}}, \chi)$$

may be identified with the triple

$$\left(L^2(d\sigma), \tilde{A}, \frac{1}{t-i} \right),$$

where $\tilde{A}$ is the multiplication operator on $L^2(d\sigma)$. For any $\varphi \in L^2(d\sigma)$ the corresponding Q-function with $a = 0$ in (0.6) has the form

$$Q(\ell) = \int_{\mathbb{R}} \left(\frac{1}{t-\ell} - \frac{t}{t^2+1} \right) (t^2+1)|\varphi(t)|^2 d\sigma(t). \tag{3.2}$$

Let $d\mu$ be the finite measure defined by $d\mu(t) = d[E(t)\chi, \chi]$, so that the measure $d\sigma$ is given by $d\sigma(t) = (t^2+1)d\mu(t)$. The condition (3.1) is equivalent to the condition (2.4). According to Proposition 2.3 there is a nonnegative measurable function h, such that $0 \le h(t) \le 1$, $t \in \mathbb{R}$, with the indicated properties. Define the function $\tilde{\varphi}(t)$ by

$$\tilde{\varphi}(t) = \sqrt{\frac{h(t)}{t^2+1}}.$$

Then the identity

$$\int_{\mathbb{R}} |\tilde{\varphi}(t)|^2 d\sigma(t) = \int_{\mathbb{R}} h(t)\, d\mu(t)$$

implies that $\tilde{\varphi} \in L^2(d\sigma)$. Let $\varphi \in \mathfrak{H}$ be the element which corresponds to $\tilde{\varphi} \in L^2(d\sigma)$. Then for the function $Q(\ell)$ (with $\mu = i$) associated with $\varphi(t)$ in (0.6), we find

$$Q(\ell) = Q^{h d\mu}(\ell).$$

The identity (2.3) shows

$$\int_{\mathbb{R}} |t||\tilde{\varphi}(t)|^2 d\sigma(t) = \int_{\mathbb{R}} |t| h(t)\, d\mu(t) = \infty.$$

This implies $\tilde{\varphi} \in L^2(d\sigma) \setminus \operatorname{dom} |\tilde{A}|^{\frac{1}{2}}$, so that $\varphi \in \mathfrak{K} \setminus \operatorname{dom} |A|^{\frac{1}{2}}$. Moreover, when $y \to \infty$, $Q(iy)$ converges to a real number. □

Theorem 3.2. *Let A be a selfadjoint operator. Then A is semibounded if and only if each one-dimensional symmetric restriction S of A is of category $\mathbf{N}_1$.*

Proof. Let A be a semibounded selfadjoint operator. Since each symmetric restriction S of A is also semibounded, it is clear that S is of category $\mathbf{N}_1$, see [2].

For the converse we have to show that A is semibounded if each one-dimensional symmetric restriction S of A is of category $\mathbf{N}_1$. Assume that A is not semibounded. According to Proposition 3.1 there exists an element $\varphi \in \mathfrak{H} \setminus \operatorname{dom} |A|^{\frac{1}{2}}$ and the Q-function $Q(\ell)$ of A and S satisfies $\lim_{y\to\infty} Q(iy) \in \mathbb{R}$. In particular, this implies that $Q(\ell)$ in (0.6) belongs to $\mathbf{N} \setminus \mathbf{N}_1$. If S is of category $\mathbf{N}_1$, this means that the

Q-function $Q(\ell)$ is exceptional, so that $Q(iy) \to \infty$ as $y \to \infty$. This contradicts $\lim_{y\to\infty} Q(iy) \in \mathbb{R}$. We conclude that S is not of category $\mathbf{N}_1$. □

Theorem 3.3. *Let A be a selfadjoint operator. Then A is bounded if and only if each one-dimensional symmetric restriction S of A is of category $\mathbf{N}_0$.*

Proof. Let A be bounded. Then each φ in (0.1) belongs to $\operatorname{dom} A$, so that $Q(\ell)$ in (0.6) belongs to $\mathbf{N}_0$. Hence, S is of category $\mathbf{N}_0$.

Conversely, let S be of category $\mathbf{N}_0$. Hence, the Q-function $Q(\ell)$ in (0.6) of S and A belongs to $\mathbf{N}_0$ (as it clearly is not exceptional, since A is an operator). Equivalently, $\varphi \in \operatorname{dom} A$. Therefore $\mathfrak{H} \subset \operatorname{dom} A$, from which it follows that A is bounded. □

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