PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 9, September 1997, Pages 2725–2732
S 0002-9939(97)04061-6

# AN EXTENSION OF THE RABINOWITZ BIFURCATION THEOREM TO LIPSCHITZ POTENTIAL OPERATORS IN HILBERT SPACES

ALEXANDER IOFFE AND EFIM SCHWARTZMAN

(Communicated by Hal L. Smith)

ABSTRACT. The main result of the paper is an extension of the bifurcation theorem of Rabinowitz to equations $Ax + \varphi_\lambda(x) = \lambda x$ with $\varphi$ continuous jointly in $(\lambda, x)$ and $\varphi_\lambda(\cdot)$ of class $C^{1,1}$. We also prove a bifurcation theorem for critical points of the function $g_\lambda(x)$ which is just continuous and changes at $x = 0$ an isolated minimum (in $x$) to isolated maximum when $\lambda$ passes, say, zero. The proofs of the theorems, as well as the the theorems themselves, are new, in certain important aspects, even when applied to smooth functions.

## 1. Introduction

In what follows $H$ is a real Hilbert space, $B_H(\alpha)$, or just $B(\alpha)$, is the closed ball of radius $\alpha$ around the origin in $H$ and $(\cdot|\cdot)$ is the inner product in $H$. Consider the function

$$f_\lambda(x) = (1/2)(Ax|x) + \varphi_\lambda(x)$$

on $[-\alpha, \alpha] \times B(\alpha)$ assuming that

- **$(a_1)$:** $A$ is a bounded self–adjoint operator on $H$;
- **$(a_2)$:** $\varphi(\lambda, x) = \varphi_\lambda(x)$ is a continuous function as well as its gradient $\nabla\varphi_\lambda(x)$ (with respect to $x$);
- **$(a_3)$:** $\nabla\varphi_\lambda(x)$ satisfies the Lipschitz condition with respect to $x$ with the constant not depending on $\lambda$.

We denote the spectrum of $A$ by $\sigma(A)$. Consider the equation

$$\nabla f_\lambda(x) = \lambda x. \tag{1}$$

Assuming that $x = 0$ satisfies the equation for all $\lambda$, we say that $\mu$ is a *bifurcation point* of (1) if there are sequences of $\lambda_n \to \mu$ and $x_n \to 0$, $x_n \neq 0$ such that $(\lambda_n, x_n)$ satisfies (1) for all $n$. Our primary goal is to prove the following theorem.

**Theorem 1.** *Suppose that $\mu$ is an isolated eigenvalue of $A$ of finite multiplicity and the Lipschitz constant of $\nabla\varphi$ with respect to $x$ is strictly smaller than*

$$r := \inf\{|\lambda - \mu| : \lambda \in \sigma(A), \lambda \neq \mu\}. \tag{2}$$

---

Received by the editors February 5, 1996.
1991 *Mathematics Subject Classification.* Primary 58E05, Secondary 49K99.
*Key words and phrases.* Critical point, $C^{1,1}$-function, bifurcation, modulus of regularity, Palais-Smail condition.
The research was supported by the US-Israel BSF grant 90-00455 and by the Israel Ministry of Science and Technology grant 3501–1–91 .

*Suppose further that for all $\lambda$ sufficiently close to $\mu$*

(3)

$$\nabla\varphi_\lambda(0) = 0; \quad \limsup_{x\to 0} \|x\|^{-2}(r^{-1}\|\nabla\varphi_\lambda(x)\|^2 + |\varphi_\lambda(x)|) < (1/2)|\lambda - \mu| \quad \textit{if } \lambda \neq \mu.$$

*Then $\mu$ is a bifurcation point of (1) and one of the following three possibilities holds:*

- (i) *$x = 0$ is not an isolated critical point of $f_\mu(\cdot)$;*
- (ii) *for every $\lambda$ in a neighborhood of $\mu$ there is a nontrivial solution $x(\lambda)$ of (1) converging to 0 as $\lambda \to \mu$;*
- (iii) *there is a one-sided (right or left) neighborhood of $\mu$ such that for any $\lambda \neq \mu$ in the neighborhood, (1) has at least two nontrivial solutions converging to zero as $\lambda \to \mu$.*

For $\varphi$ of class $C^2$ and not depending on $\lambda$ this is the same statement as in the classical theorem of Rabinowitz [1] which concluded a series of studies initiated by Krasnosel'ski in the 50s [2]. Rabinowitz's work was preceded by Böhme [3] and Marino [4] who showed that for all sufficiently small $\rho > 0$ there are at least two nontrivial solutions of (1) satisfying $|x| = \rho,\ |\lambda - \mu| \leq \rho$. The latter result was extended by McLeod and Turner [5] to functions $\varphi$ of class $C^{1,1}$ depending linearly on two parameters and such that the Lipschitz constant of the gradient going to zero as the parameters and $x$ go to zero. (Their paper also contains interesting examples when a Lipschitz rather than smooth perturbation appears in (1).) In all these results it was assumed that the Lipschitz constant of $\nabla\varphi$ goes to zero as $x \to 0$. The latter property is strictly stronger than the last relation in (3) even if $\varphi$ does not depend on $\lambda$.

We mention further a more recent paper by Kielhöffer [6] in which a bifurcation theorem was proved (for a $C^2$ case) for $f$ in which the operator $A$ also may depend on $\lambda$, that is to say, for $C^1$ potential operators which may depend arbitrarily on $\lambda$. However, unlike the theorem of Rabinowitz, the theorem of Kielhöffer only states the existence of a bifurcation and says nothing about the behavior of bifurcating solutions.

The novelty of this work lies not only in weakening of differentiability requirements on $f$ and the requirements on the mode of the dependence of $f$ on $\lambda$ but also in the technique we use in the proofs which is new for the smooth case as well.

As many earlier proofs, ours consists of two parts: reduction to a function of the form

$$g_\lambda(x) = (\lambda/2)\|x\|^2 + \psi(\lambda, x) \tag{4}$$

and bifurcation analysis for (4). The new technology we apply to bifurcation analysis of (4) leads to a simpler and space saving proof in a very general situation when we actually assume $\psi$ only continuous, even not Lipschitz. A suitable concept of a critical point and necessary techniques are provided by a recently developed "metric" critical point theory for continuous functions in complete metric spaces [7], [8], [9], [10]. In the heart of the techniques lies a Potential Well Theorem of [9] ( Proposition 2 below) giving a priori estimates for the size of the potential well associated with a given local minimum, which leads, either jointly with a "nonsmooth" mountain pass theorem of [7], [10], [11] or without it, to short and transparent proofs of the (i) – (iii) alternative.

The next section contains all necessary information from the metric critical point theory. Section 3 is devoted to a bifurcation analysis of $g_\lambda(x)$. In fact even a more general class of functions is considered there: basically this is the bifurcation analysis of a function whose parametric dependence contains a change from a local minimum to a local maximum at the reference critical point. The final section contains a proof of Theorem 1.

## 2. Preliminaries

Let $(X, d)$ be a complete metric space and $f(x)$ a continuous function on $X$.

**Definition 1.** We say that $x$ is a $\delta$–regular point of $f$ if there are a neighborhood $U$ of $x$ and a continuous mapping $\omega_\lambda(u) = \omega(\lambda, u) : [0,1] \times U \mapsto X$ such that for all $(\lambda, u) \in [0,1] \times U$

(i) $\omega_0(u) = u$ and $\omega_\lambda(u) \neq u$ if $\lambda > 0$;
(ii) $f(u) - f(\omega_\lambda(u)) \geq \delta \cdot d(u, \omega_\lambda(u))$.

Slightly changing the notation of [8], we denote by $|df|^+(x)$ the least upper bound of all $\delta > 0$ such that $f$ is $\delta$-regular at $x$ and set $|df|^+(x) = 0$ if $f$ is not $\delta$–regular at $x$ for any positive $\delta$. We set further $|df|(x) = \min\{|df|^+(x), |d(-f)|^+(x)\}$ and say that $x$ is a *critical point* of $f$ if $|df|(x) = 0$. Likewise, we call a sequence $\{x_n\}$ *critical* if $|df|(x_n) \to 0$. (Note that $|df|(x)$ is a lower semicontinuous function of $x$.) A local minimum (or maximum) is obviously a critical point. If $X$ is a Banach space and $f$ is strictly Fréchet differentiable at $x$ (that is, $\|h\|^{-1}\|f(u+h) - f(u) - f'(u)h\| \to 0$ as $u \to x$ and $h \to 0$), then $|df|(x)$ coincides with the norm of the derivative of $f$ at $x$. In other words, in this case the definition of a critical point reduces to the standard definition. If $f$ satisfies a Lipschitz condition near $x$, then $|df|(x)$ is not smaller than the distance from zero to $\partial f(x)$ (Clarke's generalized gradient of $f$ at $x$). Therefore in this case a necessary condition for an $x$ to be a critical point is that $0 \in \partial f(x)$.

**Definition 2** ([9])**.** Let $G \subset X$ be an open set and $z \notin G$. A positive nondecreasing function $\delta(t)$ on $(0, \infty)$ is a *modulus of regularity* of $f$ on $G$ with respect to $z$ if $|df|(x) \geq \delta(d(x, z))$ for all $x \in G$.

**Definition 3** ( [7], [9], [11])**.** We say that $f$ satisfies the *Palais–Smale condition* at the level $c$ if any critical sequence $\{x_n\}$ such that $f(x_n) \to c$ has a converging subsequence. If any critical sequence $\{x_n\}$ such that the sequence of $f(x_n)$ is bounded has a converging subsequence, then we say that $f$ satisfies the Palais–Smale condition (without indicating the level).

**Proposition 1.** *Let $(X, d)$ be a complete metric space and $f$ a continuous function on $X$ satisfying the Palais–Smale condition. Let $B$ be the closed ball of radius $r$ centered at $z$ such that $z$ is the unique critical point of $f$ in $B$. Then $f$ has a modulus of regularity with respect to $z$ defined on $G = \{x : 0 < d(x, z) < r\}$.*

It has to be also observed that the properties of being a critical point or a critical sequence, the existence of a modulus of regularity and the Palais–Smale condition are *metric* properties; that is to say, they are invariant with respect to Lipschitz homeomorphisms.

The following plays the crucial role in our proof of the bifurcation theorem.

**Proposition 2** (Potential Well Theorem [9])**.** *Suppose $X$ is a locally connected complete metric space, $z$ is a local minimum of $f$ and there are an $r > 0$ and a modulus of regularity $\delta(t)$ of $f$ with respect to $z$ defined on the set $\{x : 0 < d(x,z) < r\}$. Let $0 < \theta < (r/2)\delta(r/4)$, and let $B$ be a connected closed ball of radius $\leq r/4$ on which $f(x) \leq f(z) + \theta$. Then the inequality*

$$f(x) \geq f(z) + \int_0^{d(x,z)} \delta(t/2)dt$$

*holds for all $x \in B$.*

In case when the reference point $z$ is *not* a local minimum for $f$, we have the following *upper* estimate for the function.

**Proposition 3** ([9], Proposition 6)**.** *Suppose $(X,d)$ is a complete metric space, $f$ is a continuous function on $X$ and there is a modulus of regularity $\delta(t)$ of $f$ on the set $G = \{x : \ 0 < d(x,z) < r\}$ $(r > 0)$. If $z$ is not a local minimum of $f$, then for any $\tau \in [0,1]$*

$$\inf\{f(x) : \ d(x,z) \leq \tau r\} \leq f(z) - (\tau^2/2)r\delta(\tau r/4).$$

As a consequence of this fact, we get the property which also follows from the variational principle of Ekeland [14].

**Proposition 4.** *Let $(X,d)$ be a complete metric space and $f$ a continuous function on $X$ bounded from below. Then for any $\varepsilon > 0$ there is an $x$ such that $f(x) \leq \inf f + \varepsilon, \ |df|(x) \leq \varepsilon$.*

The last result we quote extends the mountain pass theorem of Ambrosetti–Rabinowitz to continuous functions on complete metric spaces. We state it in a simplified form sufficient for our purpose.

**Proposition 5** ([8],[11])**.** *Let $X$ be a complete metric space and $f$ a continuous function on $X$. Fix two points $u, v \in X$ and consider the collection $P$ of all continuous paths $p(t) : [0,1] \mapsto X$ joining $u$ and $v$, say such that $p(0) = u, \ p(1) = v$. Set*

$$c = \inf_{p \in P} \max_{0 \leq t \leq 1} f(p(t)).$$

*Assume further that $c > \max\{f(u), f(v)\}$ and that $f$ satisfies the Palais–Smale condition. Then $f$ has a critical point $x$ with $f(x) = c$.*

## 3. An auxiliary problem: a "pure" bifurcation theorem

According to the strategy described in the introduction we have to study the finite dimensional bifurcation problem for functions (4) with nondifferentiable perturbations $\varphi$ (see [15], p.159; [16], p. 745;[9] for the case of a differentiable perturbation function $\varphi$). We shall actually consider a more general situation of a function on an infinite dimensional space (even Banach). So let $g_\lambda(x)$ be defined on the product of the segment $[-\lambda_0, \lambda_0]$ and $B(\alpha)$ , where $\lambda_0$ and $\alpha$ are positive constants.

Assume that $x = 0$ is a critical point of $g_\lambda(\cdot)$ for all $\lambda$. We say that $\lambda = 0$ is a *bifurcation point* for $g$ if there are sequences $\lambda_n \to 0$ and $x_n \to 0, \ x_n \neq 0$ such that $x_n$ is a critical point of $g_{\lambda_n}(\cdot)$ for each $n$.

**Theorem 2.** *We assume that*

($a_4$)**:** *$g_\lambda(x)$ is defined and continuous on $[-\lambda_0,\lambda_0]\times B(\alpha)$ ;*
($a_5$)**:** *$x=0$ is a critical point of $g_0(\cdot)$ ; $g_\lambda(\cdot)$ has an isolated local minimum (maximum) at zero for every $\lambda>0$ and an isolated local maximum (minimum) at zero for every $\lambda<0$;*
($a_6$)**:** *$g(\lambda,\cdot)$ satisfies the Palais–Smale condition uniformly in $\lambda$; that is, if a sequence of $(\lambda_n,x_n)$ is such that $g_{\lambda_n}(x_n)$ is a bounded sequence and either $|dg_{\lambda_n}|^+(x_n)\to 0$ or $|d(-g_{\lambda_n})|^+(x_n)\to 0$, then $\{(\lambda_n,x_n)\}$ contains a subsequence converging to a certain $(\lambda,x)$ such that $x$ is a critical point of $g_\lambda(\cdot)$.*

*Then $\lambda=0$ is a bifurcation point of $g$. Moreover, at least one of the following three possibilities is always valid.*

(i) *$x=0$ is not an isolated critical point of $g_0(\cdot)$;*
(ii) *for every $\lambda$ of a neighborhood of zero there is a nontrivial critical point of $g_\lambda(\cdot)$ converging to zero as $\lambda\to 0$;*
(iii) *there is a one–sided (right or left) neighborhood of zero such that for every $\lambda\neq 0$ in the neighborhood there are two distinct nontrivial critical points of $g_\lambda(\cdot)$ converging to zero as $\lambda\to 0$.*

*Proof.* We can assume of course that $g_\lambda(0)\equiv 0$. Suppose (i) does not hold, that is, $x=0$ is an isolated critical point of $g_0(\cdot)$ . Then either zero is a local minimum or a local maximum of $g_0(\cdot)$ or $g_0(\cdot)$ assumes values of both signs in any neighborhood of zero. We shall show that in the last case (ii) holds while (iii) takes place in the first case.

1. Let $g_0(\cdot)$ assume values of both signs in any neighborhood of zero. Assuming that (ii) does not hold, we conclude that there are $\varepsilon>0$ and a sequence of $\lambda_n\to 0$ such that $g_{\lambda_n}(\cdot)$ does not have nontrivial critical points in $B(\varepsilon)$. Set $G(\varepsilon)=\{x: 0<\|x\|<\varepsilon\}$. We claim that there is a modulus of regularity $\delta(t)$ on $G$ w.r.t. zero common to all $g_{\lambda_n}(\cdot)$. If this were not true, we would find a $t>0$, a sequence of indices $\{n_k\}$ (not necessarily different) and a sequence $x_{n_k}\in B(\varepsilon)$, $\|x_{n_k}\|\geq t$ such that $d|g_{\lambda_{n_k}}(\cdot)|(x_{n_k})\to 0$. By ($a_6$), this means that either $g_0(\cdot)$ or a certain $g_{\lambda_n}(\cdot)$ has a critical point with $t\leq\|x\|\leq\varepsilon$ in contradiction with our assumption.

Assume that $\lambda_n>0$ for infinitely many $n$ (or for all of them which is the same). Then 0 is a local minimum of $g_{\lambda_n}(\cdot)$ by ($a_5$) (if $n$ is sufficiently large, to be precise). By the Potential Well Theorem (Proposition 2) there is $r>0$ such that for such $n$

$$g_{\lambda_n}(x)\geq\int_0^{\|x\|}\delta(t/2)dt>0\quad \text{if } 0<\|x\|\leq r$$

which by continuity implies the same inequality for $g_0(\cdot)$ in contradiction with the assumption that $g_0(\cdot)$ assumes values of both signs.

2. We turn now to the case when zero is a local minimum of $g_0(\cdot)$ . (The case of a maximum is similar, just replace $g$ by $-g$; see also the remark after the proof.) We first observe that by ($a_6$) every $g_\lambda(\cdot)$ satisfies the Palais–Smale condition. So by Propositions 1 and 2 there are $r>0$ and $\beta>0$ such that $g_0(x)>g_0(0)=0$ if $\|x\|\leq r$ and $g_0(x)\geq\beta$ if $\|x\|=r$. Choose a $\mu>0$ such that $|g_\lambda(x)-g_0(x)|<\beta/2$ if $|\lambda|<\mu$, $\|x\|\leq r$. Take a $\lambda\in(-\mu,0)$ and set

$$\beta(\lambda)=\frac{|\lambda|\beta}{2\mu},\ r(\lambda)=\inf\{\rho>0: g_\lambda(x)\geq\beta(\lambda)\ \text{ if } \|x\|=\rho\}.$$

It is clear that $r(\lambda) \to 0$ as $\lambda \to 0$. As $\lambda < 0$, zero is a local maximum of $g_\lambda(\cdot)$ by $(a_5)$ (for $\lambda < 0$ sufficiently close to zero). As every $g_\lambda(\cdot)$ satisfies the Palais–Smale condition, this together with Proposition 3 implies that the minimal value of $g_\lambda(\cdot)$ on $B(r(\lambda))$ is attained at a certain $x_1 = x_1(\lambda)$ with $\|x_1\| < r(\lambda)$ and

$$\varepsilon(\lambda) = g_\lambda(x_1) = \min\{g_\lambda(x) \ : \|x\| \le r(\lambda)\} < 0.$$

Thus we have proved that $g_\lambda(\cdot)$ has at least one nontrivial critical point. If there are more, then (iii) is proved. Otherwise $x_1$ is a unique global minimum of $g_\lambda(\cdot)$ on $B(r(\lambda))$. Consider the linear segment joining 0 and $u = -r(\lambda)x_1/\|x_1\|$. Then the lower bound of $g_\lambda$ on the segment is strictly greater than $\varepsilon(\lambda)$ since this segment does not contain $x_1$. If we consider now the collection $\mathcal{P}$ of all continuous paths $x(t) : [0,1] \mapsto B(r(\lambda))$ joining zero and $u$ (that is, such that $x(0) = 0, x(1) = u$), then we have to conclude, taking into account that zero is a strict local maximum of $g_\lambda(\cdot)$ , that

$$0 > c = \sup_{x(\cdot)\in\mathcal{P}} \min_{0\le t\le 1} g_\lambda(x(t)) > \varepsilon(\lambda).$$

(Indeed, as there is no critical point in a neighborhood of zero, the left inequality follows from Propositions 1 and 2 (applied to $-g_\lambda(\cdot)$ ) which imply that the upper bound of $g_\lambda(\cdot)$ on small spheres centered at zero must be negative.)

Consider now $B(r(\lambda))$ with the induced metric as a separate metric space. Then the mountain pass theorem of Proposition 5 implies that the restriction of $g_\lambda(\cdot)$ on $B(r(\lambda))$ must have a critical point at level $c$. The above inequality shows that this can be neither zero nor $x_1$. On the other hand, as $g_\lambda(\cdot)$ is positive on the boundary of $B(r(\lambda))$ the point must be strictly inside the ball, thus being a critical point of $g_\lambda(\cdot)$ itself. This completes the proof of the theorem.

*Remark.* It is possible to give a slightly longer proof of the second part of the theorem but using only the potential well theorem with no reference to a mountain pass theorem. Here is the proof for the case when zero is an isolated local *maximum* of $g_0(\cdot)$ and a positive open neighborhood of zero is considered.

As above, for any $\lambda \in (0,\mu)$ we find $r(\lambda)$ and $x_1(\lambda) \ne 0$ in the interior of $B(r(\lambda))$ such that $g_\lambda(\cdot)$ attains a maximum on $B(r(\lambda))$ at $x_1(\lambda)$ and $g_\lambda(x_1(\lambda)) = \varepsilon(\lambda) > 0$. In particular, every $g_\lambda(\cdot)$ assumes values of both signs in $B(r(\lambda))$. Denote by $\varepsilon'(\lambda)$ the upper bound of $g_\lambda(x)$ on the line segment joining zero and $u = -r(\lambda)\|x_1(\lambda)\|^{-1}x_1(\lambda)$. Then $\varepsilon'(\lambda) < \varepsilon(\lambda)$ if there is no other critical point of $g_\lambda(\cdot)$ in $B(r(\lambda))$. Consider the set

$$X(\lambda) = \{x \in H : \|x\| \ge r(\lambda) \quad \text{or } \|x\| < r(\lambda) \ \&\ g_\lambda(x) \ \le \max\{\varepsilon', \varepsilon/2\}\}.$$

(In other words, we cut away a neighborhood of the local maximum.) Consider the metric space $(X(\lambda), d)$ with the induced metric and observe that the regularity constants $|dg_\lambda(\cdot)\ |^+(x)$ at any point of $X(\lambda)$ with $\|x\| < r(\lambda)$ are the same when calculated in $X(\lambda)$ or in the whole of $X$, so that $g_\lambda(\cdot)$ satisfies the Palais–Smale condition on $X(\lambda)$ as well. This implies that $X(\lambda)$ is a connected space. Indeed, if $X(\lambda)$ has a component other than the main component containing zero and the exterior of $B(r(\lambda))$, then $g_\lambda(\cdot)$ must attain a minimum in that component which is a critical point of $g_\lambda(\cdot)$ on $X$ different from both zero and $x_1(\lambda)$.

The rest of the proof is similar to the proof of the first part of the theorem: suppose there is a sequence of positive $\lambda_n \to 0$ such that $x_1(\lambda)$ is the only nontrivial critical point of every $g_{\lambda_n}(\cdot)$ on $B(r)$. Then by $(a_6)$ there must be a modulus of regularity with respect to zero, common to the restrictions of all $g(\lambda_n, \cdot)$ to $X(\lambda)$

and this, by way of the potential well theorem through the fact that $g_\lambda(\cdot)$ attains a local minimum at zero if $\lambda > 0$, will lead to a contradiction with the fact that $g_0(\cdot)$ has a local maximum at zero.

## 4. Proof of Theorem 1

The first step of the proof is standard: the Lyapunov–Schmidt reduction. Let $U$ be the eigenspace of $A$ corresponding to $\mu$ and $V = U^\perp$. The equation (1) means that $x$ is a critical point of the function $F_\lambda(x) = f_\lambda(x) - (\lambda/2)\|x\|^2$. We can represent this function in the following form:

$$F_\lambda(x) = (1/2)(Tv|v) + \varphi_\lambda(u+v) - (1/2)(\lambda-\mu)(\|u\|^2 + \|v\|^2),$$

where $T$ is the restriction of $A - \mu I$ to $V$ and $x = u + v$. Let $P_U$ and $P_V$ denote the projections to $U$ and $V$ respectively. Then (1) is equivalent to the following two equations the left–hand parts of which are the $u$-component and the $v$-component of the gradient of $F_\lambda$:

$$\left.\begin{array}{l} P_U \nabla\varphi_\lambda(u+v) - (\lambda-\mu)u = 0; \\ Tv + P_V\nabla\varphi_\lambda(u+v) - (\lambda-\mu)v = 0. \end{array}\right\} \tag{5}$$

As the Lipschitz constant of $\nabla\varphi_\lambda$ is strictly smaller than $r = \|T^{-1}\|^{-1}$, the second equation can be resolved with respect to $v$ (for $\lambda$ sufficiently close to $\mu$), that is, there is a Lipschitz continuous mapping $v_\lambda(u)$ from a neighborhood of zero in $U$ into $V$ such that

$$v_\lambda(u) = -(T - (\lambda-\mu)I)^{-1} P_V \nabla\varphi_\lambda(u + v_\lambda(u)\ ), \quad v_\lambda(0) = 0. \tag{6}$$

Set

$$g_\lambda(u) = F_\lambda(u + v_\lambda(u)).$$

As by definition of $v_\lambda(u)$ the gradient of $F_\lambda$ is orthogonal to $V$ at $u + v_\lambda(u)$, it is clear that $g_\lambda$ is differentiable with the derivative

$$\nabla g_\lambda(u)\ = P_U\nabla\varphi_\lambda(u + v_\lambda(u)\ )\ - (\lambda-\mu)u$$

continuous jointly in $\lambda$ and $u$. This means that $u$ is a critical point of $g_\lambda(\cdot)$ if and only if $u + v_\lambda(u)$ is a critical point of $F_\lambda$, so the problem reduces to the analysis of critical points of $g_\lambda(\cdot)$. This is the end of the reduction step.

The short second step consists in verification that $g_\lambda(\cdot)$ satisfies the conditions of Theorem 3 in which case application of the theorem immediately completes the proof.

Replacing $\lambda - \mu$ by $\lambda$ and dividing (1) by $r$, we reduce the situation to the case $\mu = 0$. Condition $(a_4)$ of Theorem 3 is clearly valid. We see also from (5),(6) that $x = 0$ is a critical point of all $g_\lambda(\cdot)$. Furthermore, by (6)

$$\begin{array}{rcl} |(Tv_\lambda(u)\ |v_\lambda(u)\ )| & \le & \|T(T-\lambda I)^{-1}\|\|(T-\lambda I)^{-1}\|\|\nabla\varphi_\lambda(u+v_\lambda(u)\ )\ \|^2 \\ & \le & (1+O(\lambda))r^{-1}\|\nabla\varphi_\lambda(u+v_\lambda(u)\ )\ \|^2. \end{array}$$

Together with (3) this shows that

$$|(Tv_\lambda(u)\ |v_\lambda(u)\ )| + 2|\varphi_\lambda(u+v_\lambda(u)\ )\ | \le \lambda\|u+v_\lambda(u)\ \|^2$$

if $\lambda \neq 0$ and $u$ is sufficiently close to zero. The latter implies that $(a_5)$ also holds. Finally $(a_6)$ follows from the fact that the derivative of $g_\lambda(\cdot)$ is jointly continuous in $\lambda$ and $u$. This completes the proof of the theorem.

## Acknowledgement

Thanks are due to G. Katriel for useful comments and to the referee whose remarks helped to improve the text of the paper.

## References

[1] P.H. Rabinowitz, A bifurcation theorem for potential operators, J. Functional Analysis **25** (1977), 412–424. MR **57:**3928

[2] M.A. Krasnosel′ski, *Topological Methods in the Theory of Nonlinear Integral Equations*, GITTL, Moscow 1956 (Russian); English translation: Pergamon Press, Oxford–London–New York 1964. MR **28:**2414

[3] R. Böhme, Die Lösung der Verzweigungsgleichungen für Eigenwertprobleme, Math. Z. **127** (1972), 105–126. MR **47:**910

[4] A. Marino, La biforcazione nel caso variazionalle, Conf. Sem. Mat. Univ. Bari, no. 132 (1973). MR **50:**1068

[5] J.B. McLeod and R.E.L. Turner, Bifurcation of nondifferentiable operators with an application to elasticity, Arch. Rat. Mech. Anal. **63** (1976/77), 1–45. MR **57:**13611

[6] H. Kielhöffer, A bifurcation theorem for potential operators, J. Functional Analysis **77** (1988), 1–8. MR **89f:**58032

[7] M. Degiovanni and M. Marzocchi, A critical point theory for nonsmooth functionals, Ann. Math. Pura. Appl. **167** (1994), 73–100. MR **96a:**58043

[8] J.M. Corvellec, M. Degiovanni and M. Marzocchi, Deformation properties of continuous functionals and critical point theory, Topol. Methods in Nonlinear Analysis **1** (1993),151–171. MR **94c:**58026

[9] A. Ioffe and E. Schwartzman, Metric critical point theory 1. Morse regularity and homotopic stability of a minimum, J. Math. Pures et Appl., **75** (1996), 125–153. CMP 96:10

[10] A. Ioffe and E. Schwartzman, Metric critical point theory 2.Deformation techniques and characterization of critical levels of continuous functions, to appear in OT series volume dedicated to the memory of I. M. Glasman, Birkhäuser.

[11] G. Katriel, Mountain pass theorem and a global homeomorphism theorem, Ann. Inst. Henri Poincaré – Analyse Non–linéaire **11** (1994), 189–211. MR **95f:**58022

[12] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley 1983. MR **85m:**49002

[13] J.P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, Wiley Interscience 1984. MR **87a:**58002

[14] I. Ekeland, Nonconvex minimization problems, Bull. Amer. Math. Soc. **1** (1979), 443–474. MR **80h:**49007

[15] Shui–Nee Chow and J.K. Hale, *Methods of Bifurcation Theory*, Springer–Verlag, 1982. MR **84e:**58019

[16] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, vol II/B, Springer-Verlag, 1990. MR **91b:**47002

Department of Mathematics, The Technion, Haifa 32000, Israel
*E-mail address*: ioffe@math.technion.ac.il