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THE SPACE OF SUBCONTINUA OF A 2-DIMENSIONAL CONTINUUM IS INFINITE DIMENSIONAL

MICHAEL LEVIN AND YAKI STERNFELD

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ABSTRACT. Let X be a metric continuum and let $\mathcal{C}(X)$ denote the space of subcontinua of X with the Hausdorff metric. We settle a longstanding problem showing that if $\dim X = 2$ then $\dim \mathcal{C}(X) = \infty$. The special structure and properties of hereditarily indecomposable continua are applied in the proof.

1. INTRODUCTION AND PRELIMINARIES

Let X be a compact metrizable space. 2^X denotes the space of closed subsets of X endowed with the Hausdorff metric. 2^X is compact. $\mathcal{C}(X)$ is the closed subset of 2^X that consists of the subcontinua of X. In this note we prove that if $\dim X = 2$ then $\dim \mathcal{C}(X) = \infty$. This settles a longstanding open problem. (See [6] in particular p.217 and 226–227) It was known ([4], Nadler, Rogers, see [6]) that if $\dim X \geq 2$ and X is a hereditarily indecomposable continuum then $\dim \mathcal{C}(X) = \infty$. As by [1] every continuum of dim ≥ 3 contains some hereditarily indecomposable continuum of dim ≥ 2, $\dim X \geq 3$ implies $\dim \mathcal{C}(X) = \infty$, and our result improves this. (See also [3] for another result which implies $\dim \mathcal{C}(X) = \infty$ for 2-dimensional X which satisfies some additional conditions.) Our proof applies some of the ideas of the earlier proofs as well as the results of [1]. In particular our Lemma 1.3 was inspired by Theorem 7.8 of [4]. We wish to thank H. Kato for introducing this problem to us. The paper is self-contained modulo the results of [1]. In this section we present some preliminary results which may be of some independent interest, and in Section 2 we prove the main result and a stronger version. All spaces in this note are assumed to be separable metric.

Theorem 1.1. *Let X be an n-dimensional compact metric space, $n < \infty$. There exist an n-dimensional hereditarily indecomposable continuum Y and a light map f of Y into X.*

Proof. $\dim X \times I = n+1$, $I = [0,1]$. By [1] there exists an n-dimensional hereditarily indecomposable continuum $Y \subset X \times I$. Let $P : X \times I \longrightarrow X$ be the projection, and set $f = P|_Y$. f is light since a component of a fiber of f is a subcontinuum of both Y and I and hence must be a singleton. ☐

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Remark. Note that it follows that dim. type $Y \leq$ dim. type X in the sense of [2] and also that $\dim f(Y) = n$.

Definition 1.2 (see [6]). Let X be a continuum. A map $W : \mathcal{C}(X) \longrightarrow \mathbb{R}^+$ is called a Whitney map if W vanishes on the set of singletons in $\mathcal{C}(X)$ and if $A \neq B$ in $\mathcal{C}(X)$ and $A \subset B$ implies $W(A) < W(B)$.

Whitney maps always exist: if $\{f_n\}_{n=1}^{\infty}$ is a dense sequence of functions in $C(X, [0,1])$ and $W_n(A) = \mathrm{diam} f_n(A)$, $W_n : \mathcal{C}(X) \longrightarrow [0,1]$ then $W = \sum_{n=1}^{\infty} W_n/2^n$ is a Whitney map.

Let $d_n(Z)$ denote the n-dimensional degree as defined in [5] p. 105. Recall that $d_{n+1}(Z) < \epsilon$ if and only if Z admits a finite open cover of order $\leq n$ and mesh $< \epsilon$. Hence for Z compact $d_{n+1}(Z) = 0$ is equivalent to $\dim Z \leq n$, and $d_1(Z) < \epsilon$ if and only if the diameter of every component of Z is less than ϵ.

Lemma 1.3. *Let $\mathcal{K} \subset \mathcal{C}(Y)$ be a decomposition of a compact metric space Y such that* $\inf\{\mathrm{diam} K : K \in \mathcal{K}\} > 0$. *Let f be a light map of Y into some continuum X. Define $g : Y \longrightarrow \mathcal{C}(X)$ by $g(y) = f(K)$ where $y \in Y$ and K is the element of $\mathcal{K}$ which contains y. If g is continuous and if $g(Y)$ is a finite dimensional subset of $\mathcal{C}(X)$, then for every $\epsilon > 0$ there exists a closed subset Z of Y which intersects every element of $\mathcal{K}$ such that $d_1(Z) < \epsilon$.*

We prove Lemma 1.3 after making the following remarks (which are well known and are included for completeness).

Remarks. 1. A decomposition $\mathcal{K}$ is closed in $\mathcal{C}(Y)$ if and only if $\mathcal{K}$ is upper semi-continuous and the corresponding quotient map $q : Y \longrightarrow \mathcal{K}$ is open and monotone (see [5] p. 68).

2. If $\mathcal{K}$ is closed in $\mathcal{C}(Y)$ then it is compact and hence $\inf\{\mathrm{diam} K : K \in \mathcal{K}\} > 0$ if $\mathcal{K}$ does not contain singletons.

3. If $\mathcal{K}$ is closed in $\mathcal{C}(Y)$ then g is automatically continuous. Indeed, let a sequence $y_1, y_2, ... \in Y$ converge to $y_0 \in Y$ and let $y_i \in K_i \in \mathcal{K}$, $i \geq 0$. Since $\mathcal{K}$ is compact one can find a subsequence K_{i_j}, $i_j > 0$, converging in 2^Y to some $K \in \mathcal{K}$. Then $y_0 \in K$ and therefore $K = K_0$. Clearly $f(K_{i_j}) \longrightarrow f(K)$ in 2^X. So $g(y_{i_j}) = f(K_{i_j}) \longrightarrow f(K) = f(K_0) = g(y_0)$ and g is continuous.

Proof of Lemma 1.3. As $\mathcal{H} = g(Y)$ is finite dimensional, there exists an integer N (which depends on $\dim \mathcal{H}$) such that every open cover of $\mathcal{H}$ has an open refinement $\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_m$ so that each $\mathrm{cl}\mathcal{V}_i$ intersects at most N of the other $\mathrm{cl}\mathcal{V}_j$. (This is Lemma 7.7 of [4]. One way to obtain N is by embedding $\mathcal{H}$ in a Euclidean space.)

Let $\epsilon > 0$ and let $\delta > 0$ be sufficiently small such that the following hold:

(i) For $T \subset X$ with $\mathrm{diam} T \leq \delta$, $d_1(f^{-1}(T)) < \epsilon$ (such a δ exists since f is light).

(ii) If $B_1, B_2, ..., B_N$ are N subsets of X with $\mathrm{diam} B_i < 3\delta$, and $A \in \mathcal{H}$ then $A \setminus \bigcup_{i=1}^{N} B_i \neq \emptyset$. (Note that as $\inf\{\mathrm{diam} K : K \in \mathcal{K}\} > 0$ and f is light, $\lambda = \inf\{\mathrm{diam} H : H \in \mathcal{H}\} = \inf\{\mathrm{diam} f(K) : K \in \mathcal{K}\} > 0$ and we take $\delta < \lambda/3N$ which does the job since a continuum A of diameter $\geq \lambda$ cannot be covered by N sets of diameter $< \lambda/N$.)

Now let $\{\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_m\}$ be a closed cover of $\mathcal{H}$ with mesh$< \delta$ (mesh with respect to the Hausdorff metric in $\mathcal{C}(X)$) such that each $\mathrm{cl}\mathcal{V}_i$ intersects at most N of the other $\mathrm{cl}\mathcal{V}_j$. Note that

(iii) For each $1 \leq i \leq m$, for each $A \in \mathcal{V}_i$ and every $x \in A$, $B(x, \delta)$(=closed δ-ball in X with center at x) intersects every $B \in \mathcal{V}_i$ (since otherwise the Hausdorff distance between A and B would be more than δ).

Now we construct the required closed subset Z of Y. We shall construct inductively closed mutually distinct subsets W_i, $1 \le i \le m$, of Y with $d_1(W_i) < \epsilon$ such that W_i intersects every $K \in \mathcal{K}$ for which $f(K) \in \mathcal{V}_i$. Then we take $Z = \bigcup_{i=1}^m W_i$. Z intersects each $K \in \mathcal{K}$ since the $\mathcal{V}_i$ cover $\mathcal{H}$, and $d_1(Z) < \epsilon$ since $d_1(W_i) < \epsilon$ and W_i are mutually distinct closed sets.

To construct W_1 pick some $A_1 \in \mathcal{V}_1$ and $x_1 \in A_1$. Set $W_1 = f^{-1}(B(x_1,\delta)) \cap g^{-1}(\mathcal{V}_1)$. Assume that mutually disjoint sets $W_1, W_2, ..., W_{j-1}$ were constructed as $W_i = f^{-1}(B(x_i,\delta)) \cap g^{-1}(\mathcal{V}_i)$ where $x_i \in A_i \in \mathcal{V}_i$, $1 \le i \le j-1$. Let $A_j \in \mathcal{V}_j$. At most N of $\mathcal{V}_i$, $1 \le i \le j-1$, intersect $\mathcal{V}_j$. Assume these are $\mathcal{V}_{i_1}, \mathcal{V}_{i_2}, ..., \mathcal{V}_{i_N}$. By (ii) let $x_j \in A_j \setminus \bigcup_{l=1}^N B(x_{i_l}, 3\delta)$. Then $B(x_j,\delta) \cap B(x_{i_l},\delta) = \emptyset$ for all $1 \le l \le N$. Set $W_j = f^{-1}(B(x_j,\delta)) \cap g^{-1}(\mathcal{V}_j)$. Then $W_j \cap W_i = \emptyset$ for all $1 \le i \le j-1$. Indeed, if $i = i_l$ for some $1 \le l \le m$ this follows from $B(x_j,\delta) \cap B(x_{i_l},\delta) = \emptyset$ while if i does not belong to $\{i_1, i_2, ..., i_N\}$ it holds since $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$. By (i) $d_1(W_j) < \epsilon$ and by (iii) W_j intersects every $K \in \mathcal{K}$ such that $f(K) \in \mathcal{V}_j$. This concludes the proof of the lemma. □

The following simple observation will be applied in our proof.

Observation. Let Y be an n-dimensional compact space. There exist closed distinct subsets F_1 and F_2 of Y and $r > 0$ such that every closed subset L of Y which separates F_1 from F_2 must satisfy $d_{n-1}(L) \ge r$.

Proof. Let H_1 and H_2 be closed disjoint subsets of Y which cannot be separated in Y by a closed subset of dimension $\le n-2$. If each separator $L \subset Y$ between H_1 and H_2 satisfies $d_{n-1}(L) \ge 1$ then we are done. Otherwise let $L_1 \subset Y$ be a closed separator with $d_{n-1}(L_1) < 1$. Let U_1 be an open neighbourhood of L_1 in Y with $d_{n-1}(U_1) < 1$ and $U_1 \cap (H_1 \cup H_2) = \emptyset$. Then U_1 separates H_1 from H_2 and hence $Y \setminus U_1 = H_1^2 \cup H_2^2$, $H_1 \subset H_1^2$, $H_2 \subset H_2^2$ and H_i^2 closed in Y. If each separator L_2 between H_1^2 and H_2^2 satisfies $d_{n-1}(L_2) \ge 1/2$ then we are done. Otherwise we find a closed separator L_2 contained in an open separator U_2 with $\mathrm{cl}U_2 \subset U_1$ and $d_{n-1}(U_2) < 1/2$, and continue by an obvious induction. We obtain a decreasing sequence U_k with $\mathrm{cl}U_{k+1} \subset U_k$ and $d_{n-1}(U_k) < 1/k$. Then $L = \cap U_k$ is a separator between H_1 and H_2, and $d_{n-1}(L) = 0$ i.e. $\dim L \le n-2$ which is a contradiction. □

2. $\dim \mathcal{C}(X)$

Theorem 2.1. *Let X be a 2-dimensional continuum. Then* $\dim \mathcal{C}(X) = \infty$.

Proof. By Theorem 1.1 there exist a 2-dimensional hereditarily indecomposable continuum Y and a light map $f : Y \longrightarrow X$. Let $W : \mathcal{C}(Y) \longrightarrow \mathbb{R}^+$ be a Whitney map for $\mathcal{C}(Y)$. Let $0 < t \le W(Y)$. $W^{-1}(t)$ is a closed subset of $\mathcal{C}(Y)$. Since Y is hereditarily indecomposable $W^{-1}(t)$ is a decomposition of Y (see 1.78, p. 123 of [6]). As $t > 0$, $W^{-1}(t)$ does not contain singletons.

We make a remark before continuing with the proof of Theorem 2.1.

Remark. It can be shown that for sufficiently small $t > 0$, $\dim W^{-1}(t) = \infty$. (This is obtained in the proof of the fact that $\dim \mathcal{C}(Y) = \infty$ for 2-dimensional hereditarily indecomposable continuum Y. See [6].) And one is tempted to apply the map $f^* : \mathcal{C}(Y) \longrightarrow \mathcal{C}(X)$ defined by $K \in \mathcal{C}(Y) \longrightarrow f(K) \in \mathcal{C}(X)$ to show that $\dim f^*(W^{-1}(t)) = \infty$. But it seems as if f^* may fail to be light or even finite dimensional on $W^{-1}(t)$, and we need Lemma 1.3 to proceed.

Return to the proof of Theorem 2.1. We may apply Lemma 1.3 to $\mathcal{K} = W^{-1}(t)$ since $W^{-1}(t)$ is closed in $\mathcal{C}(Y)$. If $\mathcal{C}(X)$ is finite dimensional then by Lemma 1.3 for every $\epsilon > 0$ there exists a closed subset Z of Y with $d_1(Z) < \epsilon$ such that Z intersects each member of $W^{-1}(t)$. As $\dim Y = 2$, by the observation in the end of Section 1 there exist closed disjoint subsets F_1 and F_2 of Y and $r > 0$ such that every closed subset L of Y which separates F_1 from F_2 in Y must satisfy $d_1(L) \geq r$. Since $W^{-1}(0)$ is the space of singletons, it follows from compactness that there is a $t > 0$ such that for each $A \in W^{-1}(t)$, $\operatorname{diam} A < r$.

Set $\epsilon = \operatorname{dist}.(F_1, F_2)$, and let $Z \subset Y$ with $d_1(Z) < \epsilon$ be as above. As $d_1(Z) < \epsilon$ we can represent Z as $Z = Z_1 \cup Z_2 \cup ... \cup Z_m$ with Z_i mutually disjoint closed sets and $\operatorname{diam} Z_i < \epsilon$. Set $F_1^{'} = F_1 \cup (\bigcup\{Z_i : Z_i \cap F_1 \neq \emptyset\})$ and $F_2^{'} = F_2 \cup (\bigcup\{Z_i : Z_i \cap F_1 = \emptyset\})$. $F_1^{'}$ and $F_2^{'}$ are disjoint and closed. Let L be a closed subset of Y which separates $F_1^{'}$ from $F_2^{'}$. Then L also separates F_1 from F_2 and hence $d_1(L) \geq r$. Let L_1 be a component of L with $\operatorname{diam} L_1 \geq r$. As $L \cap Z = \emptyset$, $L_1 \cap Z = \emptyset$ too. $W^{-1}(t)$ is a decomposition of Y; hence there exists some $A \in W^{-1}(t)$ such that $L_1 \cap A \neq \emptyset$. As $\operatorname{diam} A < r$ and Y is hereditarily indecomposable $A \subset L_1$. But Z intersects each member of $W^{-1}(t)$, so $Z \cap A \neq \emptyset$ which contradicts $L \cap Z = \emptyset$. □

Theorem 2.1 shows that $\dim \mathcal{C}(X) = \infty$. Actually more is true; a similar but slightly more subtle argument implies the following result.

Theorem 2.2. *Let X be a 2-dimensional continuum and let $W : \mathcal{C}(X) \longrightarrow \mathbb{R}^+$ be a Whitney map. Then for all sufficiently small $t > 0$, $\dim W^{-1}(t) = \infty$.*

Proof. Let Y be a 2-dimensional hereditarily indecomposable continuum and let $f : Y \longrightarrow X$ be light as in the proof of Theorem 1.1. Let $W : \mathcal{C}(X) \longrightarrow \mathbb{R}^+$ be a Whitney map and let $0 < t < W(f(Y))$. $W^{-1}(t)$ is a closed subset of $\mathcal{C}(X)$. Note that no two elements of $W^{-1}(t)$ contain one another. Let $\mathcal{K}_0 = \{A : A \in \mathcal{C}(Y), f(A) \in W^{-1}(t)\}$. Each linearly ordered (by inclusion) subset of $\mathcal{K}_0$ has an upper bound in $\mathcal{K}_0$ (namely the closed union of the elements of the subset). Let $\mathcal{K}$ denote the set of all maximal members of $\mathcal{K}_0$.

$\mathcal{K}$ is a decomposition of Y. Indeed, if $A, B \in \mathcal{K}$ and $A \cap B \neq \emptyset$ then $A \subset B$ or $B \subset A$ (as Y is hereditarily indecomposable) and by the maximality of members of $\mathcal{K}$ we have $A = B$. Let $y \in Y$. $\mathcal{L} = \{A : A \in \mathcal{C}(Y), y \in A \subset Y\}$ is an arc in $\mathcal{C}(Y)$ with end points $\{y\}$ and Y. Then $\{f(A) : A \in \mathcal{L}\}$ is an arc in $\mathcal{C}(X)$ joining $\{x\} = f(\{y\})$ and $f(Y)$. Hence, as $0 < t < W(f(Y))$, there is some $A \in \mathcal{L}$ with $W(f(A)) = t$. It follows that there is some $B \in \mathcal{K}$ with $y \in A \subset B$ which shows that $\mathcal{K}$ is a decomposition of Y.

Note also that as f is light and $W^{-1}(t)$ does not contain singletons, $\inf\{\operatorname{diam} K : K \in \mathcal{K}\} > 0$ and that for every $r > 0$ there is $t_r > 0$ such that $0 < t \leq t_r$ implies that the diameter of each element of $\mathcal{K}$ is less than r.

To apply Lemma 1.3 we have to check that g is continuous. Indeed let a sequence $y_1, y_2, ... \in Y$ converge to $y_0 \in Y$ and let $y_i \in K_i \in \mathcal{K}$. Since $\mathcal{C}(Y)$ is compact one can find a subsequence K_{i_j}, $i_j > 0$, converging to $K \in \mathcal{C}(Y)$ in 2^Y. Then $y_0 \in K$ and as Y is hereditarily indecomposable $K \subset K_0$ or $K_0 \subset K$. It is clear that $f(K_{i_j}) \longrightarrow f(K)$ in 2^X and since $W^{-1}(t)$ is closed, $f(K) \in W^{-1}(t)$. So by the maximality of K_0, $K \subset K_0$ and since no two elements of $W^{-1}(t)$ contain one another $f(K_0) = f(K)$. Thus $g(y_{i_j}) = f(K_{i_j}) \longrightarrow f(K) = f(K_0) = g(y_0)$ which implies that g is continuous.

Clearly $g(Y) \subset W^{-1}(t)$. So assuming that $W^{-1}(t)$ is finite dimensional for some sufficiently small $t > 0$ we can apply Lemma 1.3 and obtain a contradiction as in the proof of Theorem 2.1. $\square$

Remark. Actually the decomposition $\mathcal{K}$ constructed in Theorem 2.2 and the decomposition $\mathcal{K} = W^{-1}(t)$ constructed in Theorem 2.1 are upper semicontinuous. Let $q : Y \longrightarrow \mathcal{K}$ be the corresponding quotient map and let $f^* : \mathcal{K} \longrightarrow \mathcal{C}(X)$ be defined by $K \longrightarrow f(K)$. (Clearly $g = f^* q$.) In Theorem 2.1 q is open but it seems that f^* may fail to be light or even finite dimensional. It can be shown that in Theorem 2.2 f^* is light but it seems that q may fail to be open. So assuming that $\mathcal{C}(X)$ is finite dimensional we are not able in the both cases to apply Kelley's theorem (Theorem 7.8 of [4]) to q and we need Lemma 1.3.

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Department of Mathematics, Haifa University, Mount Carmel, Haifa 31905, Israel
E-mail address: `levin@mathcs2.haifa.ac.il`

E-mail address: `yaki@mathcs2.haifa.ac.il`