

A WEIGHTED L^2 ESTIMATE FOR THE COMMUTATOR OF THE BOCHNER-RIESZ OPERATOR

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ABSTRACT. A weighted L^2 estimate with power weights is established for the maximal operator associated with the commutator of the Bochner-Riesz operator. An application of this weighted estimate is also given.

1. INTRODUCTION

We will work on \mathbb{R}^n , $n \geq 2$. Let λ and r be two positive numbers. The Bochner-Riesz operator T_λ^r is defined in terms of the Fourier transforms by

$$(1) \quad (T_\lambda^r f)^\wedge(\xi) = \left(1 - \frac{|\xi|^2}{r^2}\right)_+^\lambda \hat{f}(\xi),$$

where \hat{f} denotes the Fourier transform of f . Let k be a positive integer and b a BMO function, the k -th order commutator of T_λ^r is defined by

$$(2) \quad T_{\lambda; b, k}^r f(x) = T_\lambda^r ((b(x) - b(\cdot))^k f)(x).$$

If $\lambda \geq (n-1)/2$, a result of Shi and Sun [8] states that T_λ^r is bounded on $L^p(\mathbb{R}^n, w(x)dx)$ provided that $1 < p < \infty$ and $w \in A_p$, where A_p denotes the weight function class of Muckenhoupt (see [9, Chapter V] for definition and properties of A_p). Therefore, by the well-known boundedness criterion for the commutators of linear operators, which was obtained by Alvarez, Babgy, Kurtz and Pérez (see [1, Theorem 2.13]), we see that in this case $T_{\lambda; b, k}^r$ is also bounded on $L^p(\mathbb{R}^n, w(x)dx)$ for all $1 < p < \infty$ and $w \in A_p$. On the other hand, as was pointed by Hu and Lu [6], if $0 < \lambda < (n-1)/2$, the boundedness criterion of Alvarez-Babgy-Kurtz-Pérez does not apply to $T_{\lambda; b, k}^r$. Hu and Lu [6] showed that for the special case $n = 2$ or the case $\lambda > (n-1)/2(n+1)$, $T_{\lambda; b, k}^r$ enjoys some L^p mapping properties which are parallel to that of the operator T_λ^r . Furthermore, Hu and Lu [7] also considered the maximal operator associated with the commutator of the Bochner-Riesz operator, defined by

$$(3) \quad T_{\lambda; b, k}^* f(x) = \sup_{r>0} |T_{\lambda; b, k}^r f(x)|,$$

and obtained the following result.

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Theorem A. *Let k be a positive integer and b belong to $\text{BMO}(\mathbb{R}^n)$. If $\lambda > 0$, then $T_{\lambda; b, k}^*$ is bounded on $L^2(\mathbb{R}^n)$ with bound $C(n, k, \lambda)\|b\|_{\text{BMO}}^k$.*

The purpose of this paper is to establish a weighted L^2 estimate with power weights for the operator $T_{\lambda; b, k}^*$. Our main result is

Theorem 1. *Let k be a positive integer and $b \in \text{BMO}(\mathbb{R}^n)$. If $\lambda > 0$ and $0 < \alpha < 1 + 2\lambda < n$, then*

$$\int_{\mathbb{R}^n} |T_{\lambda; b, k}^* f(x)|^2 |x|^{-\alpha} dx \leq C(n, k, \lambda, \alpha) \|b\|_{\text{BMO}}^{2k} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx.$$

This estimate is of interest since it implies

Theorem 2. *Let k be a positive integer and $b \in \text{BMO}(\mathbb{R}^n)$. If $f \in L^p(\mathbb{R}^n)$ for some p with $2 \leq p < 2n/(n - 1 - 2\lambda)$, then $\lim_{r \rightarrow \infty} T_{\lambda; b, k}^r f(x) = 0$ almost everywhere.*

2. PROOF OF THEOREMS

We begin with some lemmas which will be used in the proof of Theorem 1.

Lemma 1 (see [5]). *Let k be a positive integer and $b \in \text{BMO}(\mathbb{R}^n)$. Denote by $M_{b, k}$ the k -th order commutator of the Hardy-Littlewood maximal operator defined by*

$$M_{b, k} f(x) = \sup_{r > 0} r^{-n} \int_{|x-y| < r} |b(x) - b(y)|^k |f(y)| dy.$$

If $1 < p < \infty$ and $w \in A_p$, then $M_{b, k}$ is bounded on $L^p(\mathbb{R}^n, w(x)dx)$ with bound $C(n, k, p)\|b\|_{\text{BMO}}^k$.

Lemma 2 (see [7]). *Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that $\text{supp } \varphi \subset \{1/4 \leq |x| \leq 4\}$, φ is identically one on $\{1/2 \leq |x| \leq 2\}$ and*

$$\sum_{l \in \mathbb{Z}} \varphi(2^{-l}x) = 1, \quad |x| > 0.$$

Denote by g_l the multiplier operator

$$(g_l f)^\wedge(\xi) = \varphi(2^{-l}\xi) \hat{f}(\xi).$$

Then for any positive integer k and $b \in \text{BMO}(\mathbb{R}^n)$, the k -th order commutator defined by

$$g_{l; b, k} f(x) = g_l \left((b(x) - b(\cdot))^k f \right) (x)$$

satisfies

$$\left\| \left(\sum_{l \in \mathbb{Z}} |g_{l; b, k} f|^2 \right)^{1/2} \right\|_{p, w} \leq C \|b\|_{\text{BMO}}^k \|f\|_{p, w},$$

provided that $1 < p < \infty$ and $w \in A_p$.

Lemma 3. *Let $0 < \delta < 1/2$, m_δ be a radial C^∞ function supported in $\{1 - \delta \leq |\xi| \leq 1\}$ and such that*

$$0 \leq m_\delta(\xi) \leq 1, \quad |D^\nu m_\delta(\xi)| \leq C\delta^{-|\nu|} \text{ for all multi-index } \nu.$$

Let S_δ^t be the multiplier operator defined by

$$(S_\delta^t f)^\wedge(\xi) = m_\delta(t\xi) \hat{f}(\xi).$$

For k a positive integer and $b \in \text{BMO}(\mathbb{R}^n)$, denote by $S_{\delta; b, k}^t$ the k -th order commutator of S_{δ}^t . Then for any $1 < \alpha < n$ and $\beta > 0$, there exists a positive constant $C = C(n, k, \alpha, \beta)$ such that

$$\int_1^2 \int_{\mathbb{R}^n} |S_{\delta; b, k}^t f(x)|^2 |x|^{-\alpha} dx \frac{dt}{t} \leq C \|b\|_{\text{BMO}}^{2k} \delta^{2-\alpha-\beta} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx.$$

Proof. Let $\psi_0, \psi \in C_0^\infty(\mathbb{R}^n)$ be radial functions such that

$$\text{supp} \psi \subset \{1/4 \leq |x| \leq 4\}$$

and that

$$\psi_0(x) + \sum_{l=1}^\infty \psi(2^{-l}x) = 1, \text{ if } |x| > 0.$$

Set $\psi^l(x) = \psi(2^{-l}x)$ for $l \geq 1$ and $K_{\delta}^t(x) = (m_{\delta}(t \cdot))^{\wedge}(x)$. Write

$$K_{\delta}^t(x) = K_{\delta}^t(x)\psi_0(\delta x) + \sum_{l=1}^\infty K_{\delta}^t(x)\psi^l(\delta x) = \sum_{l=0}^\infty K_{\delta}^{t,l}(x).$$

It is easy to see that m_{δ} is integrable on $L^1(\mathbb{R}^n)$ with integral less than or equal to a constant independent of δ . So if $1 \leq t \leq 2$, then

$$\|K_{\delta}^{t,l}\|_{\infty} \leq C.$$

Let $S_{\delta}^{t,l}$ be the convolution operator whose kernel is $K_{\delta}^{t,l}$. By Young's inequality, we have

$$\|S_{\delta}^{t,l} f\|_{\infty} \leq C \|f\|_1, \quad 1 \leq t \leq 2,$$

which in turn gives that

$$(4) \quad \left\| \left(\int_1^2 |S_{\delta}^{t,l} f|^2 \frac{dt}{t} \right)^{1/2} \right\|_{\infty} \leq C \|f\|_1.$$

Let $1 < \gamma < n$. The observation of Carbery, Rubio de Francia and Vega [3] tells us that

$$\int_{\mathbb{R}^n} |S_{\delta}^{1,l} f(x)|^2 |x|^{-\gamma} dx \leq C 2^{-l} \delta^{1-\gamma} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\gamma} dx.$$

This by homogeneity implies that the inequality

$$\int_{\mathbb{R}^n} |S_{\delta}^{t,l} f(x)|^2 |x|^{-\gamma} dx \leq C 2^{-l} \delta^{1-\gamma} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\gamma} dx$$

holds uniformly in $1 \leq t \leq 2$. By the same argument as in [3, page 516], it follows from the last inequality that

$$(5) \quad \int_1^2 \int_{\mathbb{R}^n} |S_{\delta}^{t,l} f(x)|^2 |x|^{-\gamma} dx \frac{dt}{t} \leq C 2^{-l} \delta^{2-\gamma} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\gamma} dx.$$

Using the interpolation theorem of Stein-Weiss (see [2, page 120]) we can obtain from inequalities (4) and (5) that for each q with $1 < q < 2$,

$$(6) \quad \int_{\mathbb{R}^n} \left(\int_1^2 |S_{\delta}^{t,l} f(x)|^2 \frac{dt}{t} \right)^{q'/2} |x|^{-\gamma} dx \leq C 2^{-l} \delta^{2-\gamma} \left(\int_{\mathbb{R}^n} |f(x)|^q |x|^{-\gamma(q-1)} dx \right)^{q'/q},$$

where $q' = q/(q - 1)$.

Now let $1 < \alpha < n$. We turn our attention to $S_{\delta; b, k}^{t, l}$, the k -th order commutator of $S_{\delta}^{t, l}$. We decompose \mathbb{R}^n into a grid of non-overlapping cubes with side length $2^l \delta^{-1}$, $\mathbb{R}^n = \bigcup_j Q_j$. Let χ_{Q_j} be the characteristic function of Q_j . Set $f_j = f \chi_{Q_j}$. Then

$$f = \sum_j f_j.$$

Since $\text{supp } K_{\delta}^{t, l} \subset \{|x| \leq C2^l \delta^{-1}\}$ when $1 \leq t \leq 2$, it is obvious that the support of $S_{\delta}^{t, l} f_j$ is contained in a fixed multiple of Q_j , and that the supports of the various terms $S_{\delta; b, k}^{t, l} f_j$ have bounded overlaps. So we have the almost orthogonality property:

$$\int_1^2 \int_{\mathbb{R}^n} |S_{\delta; b, k}^{t, l} f(x)|^2 |x|^{-\alpha} dx \frac{dt}{t} \leq C \sum_j \int_1^2 \int_{\mathbb{R}^n} |S_{\delta; b, k}^{t, l} f_j(x)|^2 |x|^{-\alpha} dx \frac{dt}{t}.$$

Thus we may assume that $\text{supp } f \subset Q$ for some cube Q with side length $2^l \delta^{-1}$. Choose $\eta \in C_0^{\infty}(\mathbb{R}^n)$, $0 \leq \eta \leq 1$, η is identically one on $50nQ$ and vanishes outside $100nQ$. Set $\bar{Q} = 200nQ$, and $\bar{b}(x) = (b(x) - b_{\bar{Q}})\eta(x)$, where $b_{\bar{Q}}$ is the mean value of b on \bar{Q} , i.e., $b_{\bar{Q}} = |\bar{Q}|^{-1} \int_{\bar{Q}} b(y) dy$. It is not difficult to see that

$$S_{\delta; b, k}^{t, l} f(x) = \sum_{m=0}^k C_k^m \bar{b}^m(x) S_{\delta}^{t, l} (\bar{b}^{k-m} f)(x).$$

Let $2 < q_1, q_2 < \infty$ such that $1/q_1 + 1/q_2 = 1/2$ and $\alpha q_2 < 2n$. By Hölder's inequality and (6), we can get

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_1^2 \left| \bar{b}^m(x) S_{\delta}^{t, l} (\bar{b}^{k-m} f)(x) \right|^2 \frac{dt}{t} |x|^{-\alpha} dx \\ & \leq \|\bar{b}^m\|_{q_1}^2 \left(\int_{\mathbb{R}^n} \left(\int_1^2 \left| S_{\delta}^{t, l} (\bar{b}^{k-m} f)(x) \right|^2 \frac{dt}{t} \right)^{q_2/2} |x|^{-\alpha q_2/2} dx \right)^{2/q_2} \\ & \leq C \left(2^{-l} \delta^{2-\alpha q_2/2} \right)^{2/q_2} \|\bar{b}^m\|_{q_1}^2 \left(\int_{\mathbb{R}^n} \left| \bar{b}^{k-m}(x) f(x) \right|^{q_2'} |x|^{-\alpha(q_2'-1)q_2/2} dx \right)^{2/q_2'} \\ & \leq C \left(2^{-l} \delta^{2-\alpha q_2/2} \right)^{2/q_2} \|\bar{b}^m\|_{q_1}^2 \|\bar{b}^{k-m}\|_{2q_2/(q_2-2)}^2 \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx \\ & \leq C \left(2^{-l} \delta^{2-\alpha q_2/2} \right)^{2/q_2} (2^l \delta^{-1})^{2n(1-2/q_2)} \|b\|_{\text{BMO}}^{2k} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx, \end{aligned}$$

where in the last inequality we have invoked the fact that (see [9, Chapter IV])

$$\|\bar{b}^m\|_{q_1} \leq C(q_1, m) \|b\|_{\text{BMO}}^m |Q|^{1/q_1}.$$

For each fixed $1 < \alpha < n$ and $\beta > 0$, we choose q_2 such that

$$2 < q_2 < \min \left\{ 2 + 1/n, 2n/\alpha, 4(1+n) / \left(2(1+n) - \beta \right) \right\}.$$

Then we have that for some positive constant ε ,

$$\int_{\mathbb{R}^n} \int_1^2 |S_{\delta; b, k}^{t, l} f(x)|^2 \frac{dt}{t} |x|^{-\alpha} dx \leq C 2^{-\varepsilon l} \delta^{2-\alpha-\beta} \|b\|_{\text{BMO}}^{2k} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx.$$

The summation of the last inequality over all $l \geq 0$ then finishes the proof of Lemma 3. □

Proof of Theorem 1. By Theorem A and the interpolation theorem of Stein-Weiss, it suffices to prove Theorem 1 for the case of $1 < \alpha < 1 + 2\lambda$. For each fixed $\delta > 0$, let m_δ and S_δ^t be the same as in Lemma 3. Set

$$S_{\delta; b, k}^* f(x) = \sup_{t>0} |S_{\delta; b, k}^t f(x)|.$$

As in [3] and [4], we have the following decomposition:

$$(1 - |\xi|^2)_+^\lambda = \sum_{i=0}^\infty 2^{-i\lambda} m_{2^{-i}}(|\xi|).$$

Therefore

$$T_{\lambda; b, k}^* f(x) \leq C \sum_{i=0}^\infty 2^{-i\lambda} S_{2^{-i}; b, k}^* f(x).$$

Clearly, if $i = 0, 1$, then

$$S_{2^{-i}; b, k}^* f(x) \leq C M_{b, k} f(x),$$

where $M_{b, k}$ is the k -th order commutator of the Hardy-Littlewood maximal operator. Recalling that $|x|^{-\gamma} \in A_2$ if and only if $-n < \gamma < n$, we thus have that by Lemma 1

$$(7) \quad \int_{\mathbb{R}^n} |S_{2^{-i}; b, k}^* f(x)|^2 |x|^{-\alpha} dx \leq C \|b\|_{\text{BMO}}^{2k} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx, \quad i = 0, 1.$$

Following along the same line as in [3], we now introduce the quadric operators

$$G_\delta f(x) = \left(\int_0^\infty |S_{\delta; b, k}^t f(x)|^2 \frac{dt}{t} \right)^{1/2}$$

and

$$\tilde{G}_\delta f(x) = \left(\int_0^\infty |\tilde{S}_{\delta; b, k}^t f(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where $\tilde{S}_{\delta; b, k}^t$ is the k -th order commutator of the following multiplier operator,

$$(\tilde{S}_\delta^t f)^\wedge(\xi) = \tilde{m}_\delta(t\xi) \hat{f}(\xi),$$

and

$$\tilde{m}_\delta(\xi) = \tilde{m}_\delta(|\xi|) = \delta |\xi| \left(\frac{d}{dr} m_\delta(r) \right) (|\xi|).$$

We want to show that if $1 < \alpha < n$, $\beta > 0$ and $0 < \delta < 1/2$, then

$$(8) \quad \int_{\mathbb{R}^n} |G_\delta f(x)|^2 |x|^{-\alpha} dx \leq C(n, k, \alpha, \beta) \delta^{2-\alpha-\beta} \|b\|_{\text{BMO}}^{2k} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx.$$

If we can do this, we can finish the proof of Theorem 1. In fact, it is easy to see that \tilde{m}_δ enjoys the same properties as those of m_δ ; thus \tilde{G}_δ satisfies the same estimate (8), as G_δ . By Schwartz's inequality, straightforward computation yields that

$$(S_{2^{-i}; b, k}^* f(x))^2 \leq C 2^i G_{2^{-i}} f(x) \tilde{G}_{2^{-i}} f(x).$$

Consequently, if $i \geq 2$,

$$(9) \quad \int_{\mathbb{R}^n} |S_{2^{-i}; b, k}^* f(x)|^2 |x|^{-\alpha} dx \leq C 2^{i(\alpha+\beta-1)} \|b\|_{\text{BMO}}^{2k} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx.$$

For each fixed $\lambda > 0$ and $1 < \alpha < 1 + 2\lambda$, we choose $\beta > 0$ such that $1 < \alpha + \beta < 1 + 2\lambda$. Combining the inequalities (7) and (9) establishes our desired estimate.

The proof of Theorem 1 is now reduced to showing (8). Observe that if $b \in \text{BMO}(\mathbb{R}^n)$, then for any $t > 0$, $b_t(x) = b(tx)$ also belongs to $\text{BMO}(\mathbb{R}^n)$ and $\|b_t\|_{\text{BMO}} = \|b\|_{\text{BMO}}$. By dilation-invariance, it follows from Lemma 3 that, for any $l \in \mathbb{Z}$,

$$(10) \quad \int_{\mathbb{R}^n} \int_{2^{-l}}^{2^{-l+1}} |S_{\delta; b, k}^t f(x)|^2 \frac{dt}{t} |x|^{-\alpha} dx \leq C \delta^{2-\alpha-\beta} \|b\|_{\text{BMO}}^{2k} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx.$$

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ as in Lemma 2. Write

$$\begin{aligned} S_{\delta; b, k}^t f(x) &= \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} \left(\varphi(2^{-l} \cdot) m_\delta(t \cdot) \right)^\wedge (x - y) (b(x) - b(y))^k f(y) dy \\ &= \sum_{l \in \mathbb{Z}} (S_\delta^t g_l)_{b, k} f(x), \end{aligned}$$

where g_l is the multiplier operator defined in Lemma 2. With the aid of the formula

$$(b(x) - b(y))^k = \sum_{m=0}^k C_k^m (b(x) - b(z))^m (b(z) - b(y))^{k-m}, \quad z \in \mathbb{R}^n,$$

we have that

$$(S_\delta^t g_l)_{b, k} f(x) = \sum_{m=0}^k C_k^m S_{\delta; b, m}^t (g_l; b, k-m f)(x).$$

Note that for each fixed t , the number of l 's for which $\text{supp}(\varphi(2^{-l} \cdot)) \cap \text{supp} m_\delta(t \cdot)$ is non-empty is at most 100. Thus

$$\begin{aligned} \int_0^\infty |S_{\delta; b, k}^t f(x)|^2 \frac{dt}{t} &\leq C \sum_{l \in \mathbb{Z}} \int_0^\infty \left| (S_\delta^t g_l)_{b, k} f(x) \right|^2 \frac{dt}{t} \\ &\leq C \sum_{m=0}^k \sum_{l \in \mathbb{Z}} \int_0^\infty \left| S_{\delta; b, m}^t (g_l; b, k-m f)(x) \right|^2 \frac{dt}{t} \\ &\leq C \sum_{m=0}^k \sum_{l \in \mathbb{Z}} \int_{2^{-l}}^{2^{-l+1}} \left| S_{\delta; b, m}^t (g_\delta; b, k-m f)(x) \right|^2 \frac{dt}{t}. \end{aligned}$$

By Lemma 2 and the inequality (10), we finally obtain that

$$\begin{aligned} \int_{\mathbb{R}^n} |G_\delta f(x)|^2 |x|^{-\alpha} dx &\leq C \sum_{m=0}^k \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} \int_{2^{-l}}^{2^{-l+1}} \left| S_{\delta; b, m}^t (g_l; b, k-m f)(x) \right|^2 \frac{dt}{t} |x|^{-\alpha} dx \\ &\leq C \delta^{2-\alpha-\beta} \sum_{m=0}^k \|b\|_{\text{BMO}}^{2m} \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} |g_l; b, k-m f(x)|^2 |x|^{-\alpha} dx \\ &\leq C \delta^{2-\alpha-\beta} \|b\|_{\text{BMO}}^{2k} \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} dx. \end{aligned}$$

□

Proof of Theorem 2. Note that if $f \in C_0^\infty(\mathbb{R}^n)$ and $\text{supp } f \subset Q$ for some cube Q , then $(b - b_Q)^m f \in L^2(\mathbb{R}^n)$ for any positive integer m . Write

$$T_{\lambda; b, k}^r f(x) = \sum_{m=0}^k C_k^m (b(x) - b_Q)^m T_\lambda^r \left((b_Q - b(\cdot))^{k-m} f \right) (x).$$

By the almost everywhere convergence of the Bochner-Riesz means (see [9, Chapter IX]), we have

$$\begin{aligned} \lim_{r \rightarrow \infty} T_{\lambda; b, k}^r f(x) &= \sum_{m=0}^k C_k^m (b(x) - b_Q)^m \lim_{r \rightarrow \infty} T_\lambda^r \left((b_Q - b(\cdot))^{k-m} f \right) (x) \\ &= \sum_{m=0}^k C_k^m (b(x) - b_Q)^m (b_Q - b(x))^{k-m} f(x) = 0. \end{aligned}$$

For each given p with $2 \leq p < 2n/(n - 1 - 2\lambda)$, we can choose α such that

$$n(1 - 2/p) < \alpha < 1 + 2\lambda.$$

It is easy to see that $L^p(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) + L^2(\mathbb{R}^n, |x|^{-\alpha} dx)$. Thus Theorem 2 follows from Theorem A and Theorem 1. \square

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