

## WEAK TYPE BOUNDS FOR A CLASS OF ROUGH OPERATORS WITH POWER WEIGHTS

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ABSTRACT. In this note we show that  $T_{\Omega,\alpha}$  and  $M_{\Omega,\alpha}$ , the fractional integral and maximal operators with rough kernel respectively, are bounded operators from  $L^1(|x|^{\beta(n-\alpha)/n}, \mathbb{R}^n)$  to  $L^{n/(n-\alpha),\infty}(|x|^\beta, \mathbb{R}^n)$ , where  $0 < \alpha < n$  and  $-1 < \beta < 0$ .

### §1. INTRODUCTION

Suppose that  $0 < \alpha < n$ , and  $\Omega \in L^s(S^{n-1})$  ( $s \geq 1$ ), where  $S^{n-1}$  denotes the unit sphere of  $\mathbb{R}^n$ . Moreover,  $\Omega$  is homogeneous of degree zero. We define the fractional maximal operator by

$$M_{\Omega,\alpha}f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|x-y|<r} |\Omega(x-y)||f(y)| dy$$

and the fractional integral operator by

$$T_{\Omega,\alpha}f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy.$$

In 1971, B.Muckenhoupt and R.Wheeden [2] studied the weighted norm inequalities for  $T_{\Omega,\alpha}$  with power weight. On the other hand, in 1993, S.Chanillo, D.Watson and R.Wheeden [1] proved that  $T_{\Omega,\alpha}$  is of weak type  $(1, n/(n-\alpha))$  under the restriction of  $s \geq n/(n-\alpha)$ . In this note, we use the idea in [3] to extend the result about weighted weak type  $(1, n/(n-\alpha))$  for  $T_{\Omega,\alpha}$  and  $M_{\Omega,\alpha}$  to power weights.

Let us now formulate our results as follows.

**Theorem 1.** *Let  $0 < \alpha < n$ ,  $-1 < \beta < 0$ ,  $n/(n-\alpha) \leq s \leq \infty$  and  $\Omega \in L^s(S^{n-1})$ . Then  $T_{\Omega,\alpha}$  is a bounded operator from  $L^1(|x|^{\beta(n-\alpha)/n}, \mathbb{R}^n)$  to  $L^{n/(n-\alpha),\infty}(|x|^\beta, \mathbb{R}^n)$ . That is, for any  $\lambda > 0$  and any  $f \in L^1(|x|^{\beta(n-\alpha)/n}, \mathbb{R}^n)$ ,*

$$\int_{\{x:|T_{\Omega,\alpha}f(x)|>\lambda\}} |x|^\beta dx \leq C \left( \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)||x|^{\beta(n-\alpha)/n} dx \right)^{n/(n-\alpha)},$$

where  $C$  is independent of  $\lambda$  and  $f$ .

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**Theorem 2.** Let  $0 < \alpha < n$ ,  $-1 < \beta < 0$ ,  $n/(n - \alpha) \leq s \leq \infty$  and  $\Omega \in L^s(S^{n-1})$ . Then  $M_{\Omega,\alpha}$  is a bounded operator from  $L^1(|x|^{\beta(n-\alpha)/n}, \mathbb{R}^n)$  to  $L^{n/(n-\alpha),\infty}(|x|^\beta, \mathbb{R}^n)$ . That is, for any  $\lambda > 0$  and any  $f \in L^1(|x|^{\beta(n-\alpha)/n}, \mathbb{R}^n)$ ,

$$\int_{\{x: |M_{\Omega,\alpha} f(x)| > \lambda\}} |x|^\beta dx \leq C \left( \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)| |x|^{\beta(n-\alpha)/n} dx \right)^{n/(n-\alpha)},$$

where  $C$  is independent of  $\lambda$  and  $f$ .

§2. PROOF OF THE THEOREMS

Let us begin by giving some lemmas.

**Lemma 1.** Let  $q > 1$  and  $T$  be a sublinear operator satisfying for each  $a > 0$  the estimate

(2.1) 
$$\left| \left\{ \frac{a}{2} \leq |x| \leq a : |T(f\chi_{\{|x|>2a\}})(x)| > \lambda \right\} \right| \leq C \left( \frac{1}{\lambda} \int_{|y|>2a} |f(y)| \left(\frac{a}{|y|}\right)^{1/q} dy \right)^q.$$

Then, if  $T$  is of weak type  $(1, q)$ , it is also of weak type  $(L^1(|x|^{\beta/q}), L^{q,\infty}(|x|^\beta))$  for  $-1 < \beta < 0$ .

*Proof.* Given  $f$ , we now define, for each  $k \in \mathbb{Z}$ ,  $f_{k,0} = f\chi_{\{|x| \leq 2^{k+1}\}}$  and  $f_{k,1} = f - f_{k,0}$ . Then we can write, as usual,

$$|Tf(x)| \leq \sum_k |Tf_{k,0}| \chi_{I_k} + \sum_k |Tf_{k,1}| \chi_{I_k} = T_0 f(x) + T_1 f(x),$$

where  $I_k = \{x \in \mathbb{R}^n : 2^{k-1} \leq |x| < 2^k\}$  for each  $k \in \mathbb{Z}$ .

If we call  $\omega_\beta(x) = |x|^\beta$ , we have

$$\begin{aligned} \omega_\beta \{x : T_0 f > \lambda\} &= \int_{\{x: T_0 f > \lambda\}} \omega_\beta(x) dx \\ &\leq C \sum_k \omega_\beta(2^k) |\{x \in I_k : T_0 f_{k,0} > \lambda\}| \\ &\leq \frac{C}{\lambda^q} \sum_k \omega_\beta(2^k) \left( \int_{|y| \leq 2^{k+1}} |f(y)| dy \right)^q \\ &= \frac{C}{\lambda^q} \sum_k \omega_\beta(2^k) \left( \sum_{j \leq k+1} \int_{I_j} |f(y)| dy \right)^q \\ &\leq \frac{C}{\lambda^q} \left\{ \sum_j \left[ \sum_{k \geq j-1} \left( \int_{I_j} |f(y)| dy \right)^q \omega_\beta(2^k) \right]^{1/q} \right\}^q \\ &= \frac{C}{\lambda^q} \left\{ \sum_j \int_{I_j} |f(y)| dy \left[ \sum_{k \geq j-1} \omega_\beta(2^k) \right]^{1/q} \right\}^q \\ &\leq \frac{C}{\lambda^q} \left\{ \sum_j \int_{I_j} |f(y)| \omega_\beta(2^j)^{1/q} dy \right\}^q \leq C \left\{ \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(y)| \omega_\beta(y)^{1/q} dy \right\}^q. \end{aligned}$$

Here we have used that  $T$  is a weak type  $(1, q)$  bounded operator and  $\beta < 0$ . In order to estimate  $T_1$ , we make use of (2.1):

$$\begin{aligned}
 \omega_\beta\{x : T_1 f > \lambda\} &= \int_{\{x : T_1 f > \lambda\}} \omega_\beta(x) dx \\
 &\leq C \sum_k \omega_\beta(2^k) |\{x \in I_k : T_1 f_{k,1} > \lambda\}| \\
 &= \frac{C}{\lambda^q} \sum_k \omega_\beta(2^k) \left( \int_{|y| > 2^{k+1}} |f(y)| \left(\frac{2^k}{|y|}\right)^{1/q} dy \right)^q \\
 &= \frac{C}{\lambda^q} \sum_k \omega_\beta(2^k) 2^k \left( \sum_{j \geq k} \int_{I_j} |f(y)| \left(\frac{1}{|y|}\right)^{1/q} dy \right)^q \\
 &\leq \frac{C}{\lambda^q} \left\{ \sum_j \left[ \sum_{k \leq j} \omega_\beta(2^k) 2^k \left( \int_{I_j} |f(y)| \left(\frac{1}{|y|}\right)^{1/q} dy \right)^q \right]^{1/q} \right\}^q \\
 &= \frac{C}{\lambda^q} \left\{ \sum_j \int_{I_j} |f(y)| \left(\frac{1}{|y|}\right)^{1/q} dy \left[ \sum_{k \leq j} \omega_\beta(2^k) 2^k \right]^{1/q} \right\}^q \\
 &\leq \frac{C}{\lambda^q} \left\{ \sum_j \int_{I_j} |f(y)| dy \frac{1}{2^{j/q}} \omega_\beta(2^j)^{1/q} 2^{j/q} \right\}^q \\
 &\leq C \left\{ \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(y)| \omega_\beta(y)^{1/q} dy \right\}^q,
 \end{aligned}$$

where we have used that  $\beta > -1$ . This finishes the proof of Lemma 1.  $\square$

**Lemma 2.** *Let  $0 < \alpha < n, \Omega \in L^s(S^{n-1})$  and  $s \geq 1$ . Then there is a  $C > 0$  depending only on  $n$  and  $\alpha$ , such that*

$$(2.2) \quad M_{\Omega, \alpha} f(x) \leq C T_{|\Omega|, \alpha}(|f|)(x).$$

*Proof.* Fix  $r > 0$ ; then we have

$$\begin{aligned}
 T_{|\Omega|, \alpha}(|f|)(x) &\geq \int_{|x-y| < r} \frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy \\
 (2.3) \quad &\geq \frac{1}{r^{n-\alpha}} \int_{|x-y| < r} |\Omega(x-y)| |f(y)| dy.
 \end{aligned}$$

Taking the supremum for  $r > 0$  on two sides of (2.3), we get

$$T_{|\Omega|, \alpha}(|f|)(x) \geq \sup_{r > 0} \frac{1}{r^{n-\alpha}} \int_{|x-y| < r} |\Omega(x-y)| |f(y)| dy.$$

This is (2.2).  $\square$

Now, let us turn to the proof of the theorems. In the first we consider Theorem 1. Since  $T_{\Omega, \alpha}$  is weak type  $(1, n/(n-\alpha))$ , by Lemma 1 we only need to show that

$T_{\Omega, \alpha}$  satisfies (2.1) for  $q = n/(n - \alpha)$  and any  $a > 0$ . In fact,

$$\begin{aligned}
& \left| \left\{ \frac{a}{2} \leq |x| \leq a : |T_{\Omega, \alpha}(f\chi_{\{|x|>2a\}})(x)| > \lambda \right\} \right| \\
& \leq \frac{1}{\lambda^q} \int_{|x| \leq a} |T_{\Omega, \alpha}(f\chi_{\{|x|>2a\}})(x)|^q dx \\
& = \frac{1}{\lambda^q} \int_{|x| \leq a} \left| \int_{|y|>2a} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} f(y) dy \right|^q dx \\
& \leq \frac{1}{\lambda^q} \left\{ \int_{|y|>2a} |f(y)| \left( \int_{|x| \leq a} \frac{|\Omega(x-y)|^q}{|x-y|^{(n-\alpha)q}} dx \right)^{1/q} dy \right\}^q \\
& \leq \frac{1}{\lambda^q} \left\{ \int_{|y|>2a} |f(y)| \left( \int_{|x-y| \leq a} \frac{|\Omega(x)|^q}{|x|^{(n-\alpha)q}} dx \right)^{1/q} dy \right\}^q \\
& \leq \frac{C}{\lambda^q} \left\{ \int_{|y|>2a} |f(y)| \frac{1}{|y|^{n-\alpha}} \left( \int_{|x-y| \leq a} |\Omega(x)|^q dx \right)^{1/q} dy \right\}^q \\
& \leq \frac{C}{\lambda^q} \left\{ \int_{|y|>2a} |f(y)| \frac{1}{|y|^{n-\alpha}} \left( \int_{|y|-a}^{|y|+a} \int_{S^{n-1}} |\Omega(\theta)|^q d\theta r^{n-1} dr \right)^{1/q} dy \right\}^q \\
& \leq \frac{C}{\lambda^q} \left\{ \int_{|y|>2a} |f(y)| \frac{1}{|y|^{n-\alpha}} \|\Omega\|_q (a|y|^{n-1})^{1/q} dy \right\}^q \\
& = C \|\Omega\|_q^q \left\{ \frac{1}{\lambda} \int_{|y|>2a} |f(y)| \left( \frac{a}{|y|} \right)^{1/q} dy \right\}^q,
\end{aligned}$$

where  $\|\Omega\|_q^q = \int_{S^{n-1}} |\Omega(\theta)|^q d\theta$ . Thus, the conclusion of Theorem 1 immediately follows from Lemma 1.

It is easy to see that the conclusion of Theorem 2 is a direct consequence of Theorem 1 and Lemma 2.

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