PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 10, October 1997, Pages 2981–2986
S 0002-9939(97)03946-4

REMARKS ON DIPERNA'S PAPER "CONVERGENCE OF THE VISCOSITY METHOD FOR ISENTROPIC GAS DYNAMICS"

GUI-QIANG CHEN

(Communicated by James Glimm)

Abstract. Concerns have been voiced about the correctness of certain technical points in DiPerna's paper (Comm. Math. Phys. **91** (1983), 1–30) related to the vacuum state. In this note, we provide clarifications. Our conclusion is that these concerns mainly arise from the statement of a lemma for constructing the viscous approximate solutions and some typos; however, the gap can be either fixed by correcting the statement of the lemma and the typos or bypassed by employing the finite difference methods.

In [Di], DiPerna found a global entropy solution of the isentropic Euler equations for the following exponents in the equation of state for the pressure:

$$\gamma = 1 + 2/(2m+1), \quad m \geq 2 \text{ integer}. \tag{1}$$

He divided his arguments into the following two steps.

1. Compactness framework

Assume that a sequence of approximate solutions $(\rho^\epsilon(x,t), m^\epsilon(x,t)), 0 \leq t \leq T$, satisfies:

(i). There exists a constant $C(T) > 0$, independent of $\epsilon > 0$, such that

$$0 \leq \rho^\epsilon(x,t) \leq C, \qquad |m^\epsilon(x,t)/\rho^\epsilon(x,t)| \leq C;$$

(ii). For all weak entropy pairs (η, q) of the isentropic Euler equations, the measure sequence

$$\eta(\rho^\epsilon, m^\epsilon)_t + q(\rho^\epsilon, m^\epsilon)_x \quad \text{is contained in a compact subset of } H^{-1}_{\text{loc}}(\mathbf{R} \times [0,T]).$$

If γ satisfies (1), then the sequence $(\rho^\epsilon(x,t), m^\epsilon(x,t))$ is compact in $L^1_{loc}(\mathbf{R} \times [0,T])$.

The reason for the restriction on the number γ is that, in such a case, any weak entropy function is a polynomial function of the Riemann invariants (w, z).

This is the key step in DiPerna's arguments and is also his main contribution to the compensated compactness method in this aspect.

Received by the editors May 16, 1996.
1991 *Mathematics Subject Classification.* Primary 35K55, 35L65; Secondary 76N15, 35L60, 65M06.
Key words and phrases. DiPerna's lemma, remarks, gap, fix, bypass, viscosity method, finite difference methods, isentropic Euler equations.

2. Construction of approximate solutions satisfying (i)-(ii)

There are several standard ways to construct the approximate solutions satisfying (i) and (ii) for the isentropic Euler equations of gas dynamics. DiPerna constructed such approximate solutions by the viscosity method, as the classical solutions of the parabolic system:

$$\begin{cases} \rho_t + m_x = \epsilon \rho_{xx}, \\ m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x = \epsilon m_{xx}, \\ (\rho, m)|_{t=0} = (\rho_0^\epsilon(x), m_0^\epsilon(x)). \end{cases} \tag{2}$$

The purpose of this note is to fix the gap in the construction of $(\rho^\epsilon, m^\epsilon)$ or $(\rho^\epsilon, u^\epsilon)$, where $u^\epsilon = m^\epsilon/\rho^\epsilon$ is the velocity when $\rho^\varepsilon > 0$.

DiPerna [Di] claimed that, if

$$\rho_0^\epsilon(x) \geq \delta_0 > 0, \qquad (\rho_0^\epsilon - \bar{\rho}, u_0^\epsilon - \bar{u}) \in C^2 \cap H^2(\mathbf{R}), \tag{3}$$

for some constants $\bar{\rho} > 0$ and $\bar{u}$, then, for any $T > 0$, there exist a global solution $(\rho^\epsilon, m^\epsilon)$ in $[0, T]$ and $\delta^\epsilon(T) > 0$ such that

$$\rho^\epsilon(x, t) \geq \delta^\epsilon(T) > 0. \tag{4}$$

The reason for this lower bound for the parabolic problem (2) is that it will avoid regions of vacuum, where the density vanishes. The bounds (3) and (4) thus allow the construction of the approximate solutions by a direct application of the method of invariant regions (see [Sm]). DiPerna attempted to show (4) by his Lemma 4.1 [Di, page 28]. This lemma is stated in an incorrect way. However, the gap can be fixed or bypassed:

(a). By actually correcting the statement of Lemma 4.1 and some typos in the energy estimates in [Di]. See Section 3, particularly Remarks 1 and 2.

(b). By avoiding entirely the viscosity method and constructing the approximate solutions by finite difference schemes.

3. Corrected version of DiPerna's Lemma 4.1

In [Di, page 28], DiPerna stated the following lemma to show the existence and estimates for the viscous solutions $(\rho^\epsilon, m^\epsilon)$.

Lemma 4.1 (DiPerna 1983)**.** *If $\phi(t)$ is a nonnegative function satisfying*

$$\phi(t) - \phi(s) \geq c_1 (t-s)^{1/2}, \quad \textit{if} \quad s < t < T < \infty,$$

$$\int_0^T \phi^{-\alpha}(s) ds \leq c_2, \quad \textit{for} \quad \alpha \in (0, 1),$$

then there exists a constant $c_3 = c_3(c_1, c_2, T, \alpha)$ such that

$$\phi(t) \geq c_3, \quad \forall\ t \in (0, T).$$

This lemma as stated above is incorrect. A simple counterexample is provided by $\phi(t) = (T - t)^{1/2}$. However, the lemma can be fixed as follows.

Corrected Lemma. *If $\phi(t)$ is a nonnegative measurable function satisfying*

$$\phi(t) - \phi(s) \geq -c_1 (t-s)^{1/2}, \quad c_1 > 0, \quad \textit{for} \quad s < t < T < \infty, \tag{5}$$

$$\int_0^T \phi^{-\alpha}(s) ds \leq c_2, \quad c_2 > 0, \quad \textit{for some} \quad \alpha \in [2, \infty), \tag{6}$$

and in addition

$$\phi(0) \geq \delta_0 > 0, \tag{7}$$

then there exists a positive constant $c_3 = c_3(c_1, c_2, T, \alpha, \delta_0)$ *such that*

$$\phi(t) \geq c_3 > 0, \quad \forall\, t \in [0, T). \tag{8}$$

Proof. It suffices to show (8) under the assumption that $\phi(t)$ is bounded. Otherwise, we replace $\phi(t)$ by $\tilde{\phi}(t) = \min(\phi(t), \delta_0)$ to show that $\tilde{\phi}(t)$ satisfies (8), which implies that $\phi(t)$ satisfies (8).

1. First the condition (5) implies

$$\phi(t) \geq \phi(0) - c_1 t^{1/2} > 0, \qquad \text{for} \quad 0 < t < \delta_0^2/c_1^2. \tag{9}$$

Furthermore, the condition (5) also implies

$$\phi(t) \geq \liminf_{s \to t-0} \phi(s), \qquad \liminf_{s \to t+0} \phi(s) \geq \liminf_{s \to t-0} \phi(s), \qquad t \in (0, T). \tag{10}$$

This may be seen as follows. In both sides of (5), take the inferior limit as $s \to t - 0$ to obtain

$$\phi(t) \geq \liminf_{s \to t-0} \phi(s).$$

In both sides of (5):

$$\phi(\tau) - \phi(s) \geq -c_1(\tau - s)^{1/2}, \quad c_1 > 0, \quad \text{for} \quad s < \tau < T < \infty,$$

we take the inferior limit when $s \to t - 0, s < t$, and then take the inferior limit when $\tau \to t + 0, \tau > t$. One has

$$\liminf_{\tau \to t+0} \phi(\tau) - \liminf_{s \to t-0} \phi(s) \geq 0,$$

which is equivalent to the second inequality of (10).

2. Now we prove (8) for the lemma arguing by contradiction. If the estimate (8) is not true, then there exists $t_* \in (0, T]$ from (9) and (10) such that

$$\begin{cases} \phi(t) > 0, & t \in [0, t_*), \\ \liminf\limits_{t \to t_* - 0} \phi(t) = 0. \end{cases} \tag{11}$$

Then, in both sides of (5), we take the inferior limit as $t \to t_* - 0$ to obtain

$$0 < \phi(s) \leq c_1(t_* - s)^{1/2}, \qquad 0 \leq s < t_*.$$

Then we have

$$c_1^{-\alpha} \int_0^{t_*} (t_* - s)^{-\alpha/2} ds \leq \int_0^{t_*} \phi^{-\alpha}(s) ds \leq c_2 < \infty.$$

However, the left-hand side of the above inequality is $+\infty$ because $\alpha \geq 2$, which is a contradiction. This completes the proof. □

Remark 1. The new elements in Corrected Lemma are the lower bound (7) and the specification of both the index $\alpha \geq 2$ and the constant $c_1 > 0$. Indeed the function $\phi(t)$ in [Di] is given by

$$\phi(t) \equiv \inf_{x \in \mathbf{R}} \rho^{\varepsilon}(x, t), \qquad \phi(0) \equiv \inf_{x \in \mathbf{R}} \rho_0(x).$$

Now part of the main assumption of the existence theorem is that $\rho_0(x) \geq \delta_0 > 0$, $\forall x \in \mathbf{R}$, which implies (7). In fact, the function $\phi(t)$ is bounded, which naturally follows from the method of invariant regions for the local solution $(\rho^\varepsilon, m^\epsilon)$. The lower positive bound c_3 of ρ^ϵ depends not only on c_1, c_2, T, α, but also on $\delta_0 > 0$.

Remark 2. There are several typos in the energy estimates in [Di, pages 27-29]. In particular, some $\alpha \in (0,1)$ should be some $\alpha \in [2,\infty)$ for his purpose. It can be checked that the energy estimates for $\rho^\varepsilon(x,t)$ yield

$$\int_0^T \sup_x (\rho^\varepsilon(x,s))^{-\alpha} ds \le c_2,$$

for some fixed $\alpha \in [2,\infty)$, which implies (6), where c_2 depends on $\alpha, T, \varepsilon, \gamma, \delta_0$, and the modulus of the Cauchy data.

Combining the Corrected Lemma with the standard energy estimates as in [Di, pages 26-29] yields

$$\rho^\epsilon(x,t) \ge \delta^\varepsilon(T) > 0, \qquad \text{for} \quad t \in [0,T].$$

Then the following existence theorem follows as in [Di].

Existence Theorem. *Suppose that the initial data* $(\rho_0^\varepsilon, m_0^\varepsilon)$ *satisfy the condition* (3). *Then there exists a global solution* $(\rho^\varepsilon, m^\varepsilon)$ *of the Cauchy problem* (2) *such that*

$$(\rho^\varepsilon(\cdot,t) - \bar{\rho}, u^\varepsilon(\cdot,t) - \bar{u}) \in C^2 \cap H^2(\mathbf{R}),$$

and

$$\rho^\varepsilon(x,t) \ge \delta^\varepsilon(T) > 0, \qquad \textit{for} \quad t \in [0,T],\ T < \infty,$$

for an appropriate function $\delta^\epsilon(T) > 0$.

4. Construction of the approximate solutions via the Lax-Friedrichs scheme and the Godunov scheme

As an alternative to the parabolic regularization, we ([C1], [C2], [D1], [D2], [CG]) employed a different approach: finite difference schemes, especially the Lax-Friedrichs scheme [L] and the Godunov scheme [G], for the construction of the approximate solutions. In this way, the issue of vacuum and the gap in [Di] was completely bypassed. Any result in [C1], [C2], [D1], [D2], and [CG] is irrelevant to the gap because there is no need of the estimate (4) for the finite difference schemes.

The main advantage of the Lax-Friedrichs scheme and the Godunov scheme is that the Riemann solutions are their building blocks. These in turn, for the equations of isentropic gas dynamics, are well understood even in the presence of vacuum ($\rho = 0$) (see [CH]). More precisely, one can show that, for the Riemann data

$$(\rho, m)|_{t=0} = \begin{cases} (\rho_-, m_-), & x < 0, \\ (\rho_+, m_+), & x > 0, \end{cases} \qquad \text{with} \quad 0 \le \rho_\pm \le C_0, |m_\pm/\rho_\pm| \le C_0,$$

there exists a corresponding Riemann solution satisfying

$$0 \le \rho(x,t) \le C, \qquad |m(x,t)/\rho(x,t)| \le C,$$

where C depends only on C_0 in a precise manner. The regions in the state space, where these inequalities hold, are called invariant regions for the Riemann solutions. Another advantage of these schemes is the finite speed of propagation of the corresponding approximate solutions so that the second of (3) becomes immaterial.

Since the vacuum problem is well posed in the approximate solutions from these schemes, then the existence of global entropy solutions for arbitrarily L^∞ initial data including the case of general γ, $1 < \gamma \leq 5/3$, follows from the more general compactness framework in [C1] (also see [D1]).

We stress that the possible occurrence of vacuum in the approximate solutions is naturally incorporated into the compactness framework (see (i)). In fact (i) and (ii) imply the strong compactness of $(\rho^\epsilon, m^\epsilon)$, which are the correct physical variables near the vacuum. By contrast, $(\rho^\epsilon, u^\epsilon)$ are not natural variables since the velocity u is not well defined in the vacuum, although it is usually bounded provided that the initial data have this property. This compactness framework correctly takes the vacuum into account (see [CG]).

For recent important contributions to the compensated compactness framework, see Lions-Perthame-Tadmor [LP1] for $\gamma \geq 3$ and Lions-Perthame-Souganidis [LP2] for $5/3 < \gamma < 3$. Also see Chen and LeFloch [CL] for a general pressure law.

A complete proof of the existence theorem and more detailed discussions on these remarks can be found in the forthcoming lecture notes [C3]. Also see Tartar [Ta] and Murat [Mu] for their pioneering work on the compensated compactness method and Ball [Ba] for a related observation.

Acknowledgements

The author is grateful to E. DiBenedetto and J. Glimm for their valuable suggestions and constant encouragements. The author would like to thank C. M. Dafermos, P. D. Lax, P. L. Lions, and T.-P. Liu for their encouragements, as well as H. Frid and P.-T. Kan for discussions. Gui-Qiang Chen's research was supported in part by the Office of Naval Research grant N00014-91-J-1384, the National Science Foundation grant DMS-9623203, and by an Alfred P. Sloan Foundation Fellowship.

References

[Di] DiPerna, R., *Convergence of the viscosity method for isentropic gas dynamics*, Commun. Math. Phys. **91** (1983), 1-30. MR **85i:**35118

[C1] Chen, G.-Q., *Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics (III)*, Acta Mathematica Scientia **8** (1988), 243-276; also **6** (1986), 75-120. MR **90c:**76099; MR **89f:**76007

[C2] Chen, G.-Q., *Remarks on spherically symmetric solutions to the compressible Euler equations*, Proc. Roy. Soc. Edinburgh **120A** (1997).

[C3] Chen, G.-Q., *Compactness Methods and Entropy Analysis for Conservation Laws*, Lecture Notes, In preparation.

[D1] Ding, X., Chen, G.-Q. and Luo, P., *Convergence of the Lax-Friedrichs scheme for isentropic gas dynamics (I)-(II)*, Acta Mathematica Scientia **7** (1987), 467-480, **8** (1988), 61-94; also **5** (1985), 415-432, 433-472. MR **90b:**76001; MR **90b:**76002; MR **89f:**76006

[D2] Ding, X., Chen, G.-Q. and Luo, P., *Convergence of the fractional step Lax-Friedrichs scheme and Godunov scheme for the isentropic system of gas dynamics*, Commun. Math. Phys. **121** (1989), 63-84. MR **90d:**65168

[CL] Chen, G.-Q. and LeFloch, P. G., *Isentropic Euler Equations with general pressure law*, Preprint, May 1997.

[CG] Chen, G.-Q. and Glimm, J., *Global solutions to the compressible Euler equations with geometrical structure*, Commun. Math. Phys. **180** (1996), 153–193.

[L] Lax, P., *Weak solutions of nonlinear hyperbolic equations and their numerical computation*, Comm. Pure Appl. Math. **7** (1954), 159-193. MR **16:**524g

[G] Godunov, S., *A difference method for numerical calculation of discontinuous solutions of the equations of hydrodynamics*, Mat. Sb. **47(89),** (1959), 271-360. MR **22:**10194

[GM] Glimm, J. and Majda, A., *Multidimensional Hyperbolic Problems and Computations*, Dedicated to the Contributions of R. DiPerna, IMA. **29**, Springer-Verlag: New York, 1991. MR **91h:**00031

[Ev] Evans, L. C., *Weak convergence methods for nonlinear partial differential equations*, CBMS Lecture Notes of A.M.S. 1990. MR **91a:**35009

[Fr] Frid Neto, H., *Compensated Compactness Applied to Conservation Laws*, Col'oquio 19, Brasileiro de Matemática, IMPA, Estrada Dona Castorina, 110-J. Botânico, CEP:22460.320, Rio de Janeiro-RJ.

[LP1] Lions, P. L., Perthame, B., and Tadmor, E., *Kinetic formulation of the isentropic gas dynamics and p-systems*, Commun. Math. Phys. **68** (1994), 415-431. MR **95e:**76093

[LP2] Lions, P. L., Perthame, B., and Souganidis, E., *Existence of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eulerian and Lagrangian coordinates*, Preprint, 1995.

[Sm] Smoller, J., *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983. MR **84d:**35002

[CH] Chang, T. and Hsiao, L., *The Riemann Problem and Interaction of Waves in Gas Dynamics*, Longman Scientific & Technical, Essex, England, 1989. MR **90m:**35122

[Ba] Ball, J. M., *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1977), 337-403. MR **57:**14788

[Se] Serre, D., *La compacité par compensation pour les systémes hyperboliques non linéaire de deux équations à une dimension d'espace*, J. Math. Pures Appl. **65** (1986), 423-468. MR **88d:**35123

[Ta] Tartar, L., *Compensated compactness and applications to partial differential equations*, Research Notes in Mathematics, Nonlinear Analysis, and Mechanics, ed. R. J. Knops, vol. 4, Pitman Press, New York, 1979. MR **81m:**35014

[Mu] Murat, F., *A survey on compensated compactness*, In: Contributions to Modern Calculus of Variations, Pitman Research Notes in Math. **148** (1987), 145-183.

Department of Mathematics, Northwestern University, Evanston, Illinois 60208-2730

E-mail address: `gqchen@math.nwu.edu`