

PRESCRIBING GAUSSIAN CURVATURE ON R^2

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(Communicated by Peter Li)

ABSTRACT. We derive a sufficient condition for a radially symmetric function $K(x)$ which is positive somewhere to be a conformal curvature on R^2 . In particular, we show that every nonnegative radially symmetric continuous function $K(x)$ on R^2 is a conformal curvature.

In this paper, we consider the prescribing Gaussian curvature problem. Let (M, g) be a Riemannian manifold of dimension 2 with Gaussian curvature k . Given a function K on M , one may ask the following question: Can we find a new conformal metric g_1 on M (i.e., there exists u on M such that $g_1 = e^{2u}g$) such that K is the Gaussian curvature of g_1 ? This is equivalent to the problem of solving the elliptic equation

$$(0) \quad \Delta u - k + Ke^{2u} = 0$$

on M , where Δ is the Laplacian of (M, g) . This problem has been considered by many authors. In case M is compact, we refer to [6] for details and references.

In case $M = R^2$, equation (0) becomes

$$(1) \quad \Delta u + K(x)e^{2u} = 0$$

and this problem is well understood if $K(x)$ is nonpositive; in particular, if $|K(x)|$ decays slower than $|X|^{-2}$ at infinity, then equation (1) has no solution (see [11], [13]). However, if $K(x)$ is positive at some point, the situation is totally different. If $K(x_0) > 0$ for some $x_0 \in R^2$, R. C. McOwen [10] proved that, for $K(x) = O(r^{-l})$ as $r \rightarrow \infty$, equation (1) has a C^2 solution, where l is a positive constant. Also, it is not difficult to see that equation (1) has solutions for every positive constant $K(x) = C$.

Since there is no known nonexistence result for $K \geq 0$ on R^2 , one may propose the following

Problem 1. Is it true that every nonnegative function (smooth enough) on R^2 is a conformal Gaussian curvature function?

Received by the editors May 10, 1996.

1991 *Mathematics Subject Classification.* Primary 58G30; Secondary 53C21.

Key words and phrases. Prescribing Gaussian curvature, semilinear elliptic PDE, integral equation.

We shall prove an existence theorem for equation (1) when $K(x)$ is a radially symmetric function. As usual, we set

$$\begin{aligned} K_-(x) &= \min\{K(x), 0\}, \\ K_+(x) &= \max\{K(x), 0\}, \end{aligned}$$

so $K(x) = K_-(x) + K_+(x)$.

Theorem 1. *If $K(x) = \tilde{K}(r)$ is a radially symmetric continuous function on R^2 , and there exists an $\alpha > 0$ such that*

$$(2) \quad \int_0^\infty s^{(1+2\alpha)} |\tilde{K}_-(s)| ds < \infty,$$

then equation (1) has infinitely many solutions.

Corollary 2. *If $K(x) \geq 0$ is a radially symmetric continuous function on R^2 , then equation (1) has infinitely many solutions.*

Remark 3. The above theorem seems to suggest a positive answer to Problem (1). This is particularly interesting because in dimensions $n \geq 3$, not every positive function on R^n is a conformal scalar curvature function. W. M. Ni [12] has shown that a nonnegative function $K(x)$ on R^n cannot be a conformal scalar curvature function if $K(x)$ satisfies $K(x) \geq C|x|^l$ near ∞ , where $C > 0$ and $l > 2$ are constants. Moreover, W. Y. Ding and W. M. Ni [3] have shown that there exist smooth radial functions $K(x)$ which are constant at infinity such that the equation

$$\Delta u + K(x)u^{\frac{n+2}{n-2}} = 0$$

has no radial solution.

Our proof is based on the following Schauder-Tychonoff fixed point theorem (cf. [1], [4]).

Theorem (Schauder-Tychonoff). *Let E be a separated locally convex topological vector space, let A be a nonempty closed convex subset of E , and let T be a continuous map of A into itself such that $T(A)$ is relatively compact (i.e., $\overline{T(A)}$ is compact) in E . Then T admits at least one fixed point.*

Proof of Theorem 1. Let $K(x) = \tilde{K}(r)$ with $r = |x|$; we try to find a solution $u(r)$ of (1) with $u(0) = \beta$ and $u'(0) = 0$. Then (1) is equivalent to the following integral equation:

$$(3) \quad u(r) = \beta - \int_0^r s \log\left(\frac{r}{s}\right) \tilde{K}(s) e^{2u(s)} ds.$$

Now we choose $0 < \alpha' < \alpha$ and β such that

$$(4) \quad \int_0^e s \log\left(\frac{e}{s}\right) |\tilde{K}_-(s)| e^{2(\beta+1)} ds < \frac{1}{2},$$

$$(5) \quad \int_0^e s |\tilde{K}_-(s)| e^{2(\beta+1)} ds < \frac{\alpha'}{2},$$

$$(6) \quad \int_e^\infty s^{(1+2\alpha')} |\tilde{K}_-(s)| e^{2(\beta+1)} ds < \frac{\alpha'}{2},$$

$$(7) \quad \int_e^\infty s^{(1+2\alpha')} \log\left(\frac{e}{s}\right) |\tilde{K}_-(s)| e^{2(\beta+1)} ds < \frac{1}{2}.$$

Define the functions $A_\beta(r)$ and $B_\beta(r)$ by

$$(8) \quad \begin{cases} A_\beta(r) = (\beta + 1), & \text{if } 0 \leq r \leq e, \\ A_\beta(r) = (\beta + 1) + \alpha' \log\left(\frac{r}{e}\right), & \text{if } e \leq r, \end{cases}$$

$$(9) \quad B_\beta(r) = \beta - \int_0^r s \log\left(\frac{r}{s}\right) \tilde{K}_+(s) e^{2A_\beta(s)} ds.$$

Let X denote the locally convex space of all continuous functions on $[0, \infty)$ with the usual topology, *i.e.*, $\lim_{n \rightarrow \infty} f_n = f$ in X iff f_n converges to f uniformly on any compact subset of $[0, \infty)$.

Now consider the set

$$Y = \{u \in X \mid B_\beta(r) \leq u(r) \leq A_\beta(r), \quad r \in [0, \infty)\}.$$

It is easy to see that Y is a closed convex subset of X . Let T be the mapping

$$(10) \quad (Tu)(r) = \beta - \int_0^r s \log\left(\frac{r}{s}\right) \tilde{K}(s) e^{2u(s)} ds.$$

We shall prove that T is a continuous mapping from Y into itself such that TY is relatively compact.

First, we verify that $TY \subset Y$. Assume $u \in Y$. Hence we have

$$(11) \quad B_\beta(r) \leq u(r) \leq A_\beta(r), \quad r \in [0, \infty).$$

It is easy to see that Tu is continuous. Now for $0 \leq r \leq e$ we have

$$\begin{aligned} (Tu)(r) &= \beta - \int_0^r s \log\left(\frac{r}{s}\right) \tilde{K}_-(s) e^{2u(s)} ds - \int_0^r s \log\left(\frac{r}{s}\right) \tilde{K}_+(s) e^{2u(s)} ds \\ &\leq \beta - \int_0^e s \log\left(\frac{e}{s}\right) \tilde{K}_-(s) e^{2(\beta+1)} ds \\ &\leq (\beta + 1) = A_\beta(r). \end{aligned}$$

For $e \leq r$, we have

$$\begin{aligned} (Tu)(r) &= \beta - \int_0^r s \log\left(\frac{r}{s}\right) \tilde{K}_-(s) e^{2u(s)} ds - \int_0^r s \log\left(\frac{r}{s}\right) \tilde{K}_+(s) e^{2u(s)} ds \\ &\leq \beta - \log\left(\frac{r}{e}\right) \int_0^e s \tilde{K}_-(s) e^{2(\beta+1)} ds - \int_0^e s \log\left(\frac{e}{s}\right) \tilde{K}_-(s) e^{2(\beta+1)} ds \\ &\quad - \log\left(\frac{r}{e}\right) \int_e^\infty s^{(1+2\alpha')} \tilde{K}_-(s) e^{2(\beta+1)} ds \\ &\quad - \int_e^\infty s^{(1+2\alpha')} \log\left(\frac{e}{s}\right) \tilde{K}_-(s) e^{2(\beta+1)} ds \\ &\leq \beta + \frac{\alpha'}{2} \log\left(\frac{r}{e}\right) + \frac{1}{2} + \frac{\alpha'}{2} \log\left(\frac{r}{e}\right) + \frac{1}{2} \\ &= (\beta + 1) + \alpha' \log\left(\frac{r}{e}\right) = A_\beta(r). \end{aligned}$$

On the other hand, since $u(r) \in Y$, we have

$$\begin{aligned} (Tu)(r) &= \beta - \int_0^r s \log\left(\frac{r}{s}\right) \tilde{K}_-(s) e^{2u(s)} ds - \int_0^r s \log\left(\frac{r}{s}\right) \tilde{K}_+(s) e^{2u(s)} ds \\ &\geq \beta - \int_0^r s \log\left(\frac{r}{s}\right) \tilde{K}_+(s) e^{2A_\beta(s)} ds \\ &= B_\beta(r). \end{aligned}$$

This verifies that $T Y \subset Y$.

To show that T is continuous in Y , let $\{u_m\}_{m=1}^\infty \subset Y$ be a sequence converging to $u \in Y$ in the space X . Then u_m converges to u uniformly on any compact interval of $[0, \infty)$. Now

$$(12) \quad |Tu_m(r) - Tu(r)| \leq \int_0^r s \log\left(\frac{r}{s}\right) |\tilde{K}(s)| |e^{2u_m(s)} - e^{2u(s)}| ds,$$

but

$$\begin{aligned} s \log\left(\frac{r}{s}\right) |\tilde{K}(s)| |e^{2u_m(s)} - e^{2u(s)}| &\leq s \log\left(\frac{r}{s}\right) |\tilde{K}(s)| (e^{2A_\beta(s)} - e^{2B_\beta(s)}) \\ &\leq s \log\left(\frac{r}{s}\right) |\tilde{K}(s)| e^{2A_\beta(s)} \end{aligned}$$

and $s \log(\frac{r}{s}) |\tilde{K}(s)| e^{2A_\beta(s)}$ is integrable on any compact interval of $[0, \infty)$. Hence from (12) and the uniform convergence of u_m to u on any compact interval, we conclude that Tu_m converges to Tu uniformly on any compact interval, which implies that Tu_m converges to Tu in X . This verifies that T is continuous in Y .

We can easily compute that

$$|(Tu)'(r)| = \left| \int_0^r \left(\frac{s}{r}\right) \tilde{K}(s) e^{2u(s)} ds \right| \leq \int_0^r \left(\frac{s}{r}\right) |\tilde{K}(s)| e^{2A_\beta(s)} ds.$$

Hence, on any compact interval of $[0, \infty)$, $T Y$ is uniformly bounded and equicontinuous. This proves that $T Y$ is relatively compact in Y . So by the Schauder-Tychonoff fixed point theorem, T has a fixed point u in Y . This u is a solution of (2) and hence a solution of (1). We notice that, if (3) has a solution for some β , then it has a solution for all $\beta_1 \leq \beta$. This completes the proof of Theorem 1. \square

In the $n \geq 3$ dimension case, if we also assume that $K(x) = \tilde{K}(r)$ is radially symmetric in (1), and we want to find a radially symmetric solution $u(r)$ such that $u(0) = \beta$ and $u'(0) = 0$, then (1) is equivalent to

$$(13) \quad u(r) = \beta - \frac{1}{n-2} \int_0^r s \left(1 - \left(\frac{s}{r}\right)^{n-2}\right) \tilde{K}(s) e^{2u(s)} ds.$$

In this situation we can show the following

Theorem 4. *If $K(x) = \tilde{K}(r)$ is a radially symmetric continuous function on R^n , $n \geq 3$, such that*

$$(14) \quad \int_0^\infty s |\tilde{K}_-(s)| ds < \infty,$$

then the equation

$$(15) \quad \Delta u + K(x) e^{2u} = 0$$

has infinitely many solutions.

Proof. The argument is essentially the same as in the proof of Theorem 1. We leave the details to the readers. \square

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