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THE SCHWARZ-PICK LEMMA FOR DERIVATIVES

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Abstract. The Schwarz-Pick Lemma states that any analytic function of the unit disc into itself is a contraction with respect to the hyperbolic metric. In this note a related result is proved for the derivative of an analytic function.

1. Introduction

In [2] Dieudonne showed that if $f : \mathbb{D} \to \mathbb{D}$ is analytic, where $\mathbb{D}$ is the open unit disc in the complex plane, and if $f(0) = 0$, then

$$|f'(z)| \leq \begin{cases} 1 & \text{if } |z| \leq \sqrt{2} - 1, \\ \frac{(1+|z|^2)^2}{4|z|(1-|z|^2)} & \text{if } |z| \geq \sqrt{2} - 1. \end{cases} \tag{1}$$

He then suggested (in the footnote on page 352) that this inequality, which is best possible for each value of $|z|$, should be considered as a Schwarz Lemma for the derivative f'. In this note we derive a more elegant version of this that is closer to the classical Schwarz-Pick Lemma.

As f maps $\mathbb{D}$ into itself one can argue that any discussion of f should be in terms of the hyperbolic metric rather than the Euclidean metric. The disk $\mathbb{D}$ is endowed with the hyperbolic metric ρ derived from $2|dz|/(1 - |z|^2)$, and the hyperbolic derivative $f^*(z)$ of f at z is given by

$$f^*(z) = \left(\frac{1 - |z|^2}{1 - |f(z)|^2} \right) f'(z).$$

Pick's version of Schwarz's Lemma, namely that $\rho(fz, fw) \leq \rho(z, w)$, guarantees that $|f^*(z)| < 1$ in $\mathbb{D}$ (unless f is a Möbius map of $\mathbb{D}$ onto itself) and this means that we can measure the hyperbolic distance between two hyperbolic derivatives. This leads to the following Schwarz-Pick Lemma for derivatives.

Theorem. *If $f : \mathbb{D} \to \mathbb{D}$ is analytic but not a conformal automorphism of $\mathbb{D}$, and if $f(0) = 0$, then*

$$\rho(f^*(0), f^*(z)) \leq 2\rho(0, z). \tag{2}$$

Further, equality holds in (2) *for each z when $f(z) = z^2$.*

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If $f(z) = z^2$, then $f^*(0) = 0$ and $f^*(z) = 2z/(1+|z|^2)$, and a simple calculation using the fact that

$$\rho(0,z) = \log \frac{1+|z|}{1-|z|}$$

shows that equality holds in (2).

2. The proof of the Theorem

We begin with a preliminary result.

Lemma 1. *Let z_0 and w_0 be points of D with $|w_0| < |z_0|$. If $f : \mathbb{D} \to \mathbb{D}$ is analytic, and if $f(0) = 0$ and $f(z_0) = w_0$, then both $f^*(0)$ and $f^*(z_0)$ lie in the closed hyperbolic disc*

$$D = \{z : \rho(z, w_0/z_0) \le \rho(0, z_0)\}.$$

Proof. Given z_0 and w_0, we define maps $h : \mathbb{D} \to \mathbb{D}$ and $g : \mathbb{D} \to \mathbb{D}$ by

$$h(z) = \frac{f(z)}{z}, \qquad \frac{f(z) - f(z_0)}{1 - f(z)\overline{f(z_0)}} = g(z)\left(\frac{z - z_0}{1 - z\overline{z_0}}\right). \tag{3}$$

It is immediate that

$$h(0) = f'(0) = f^*(0), \quad h(z_0) = \frac{w_0}{z_0}, \qquad g(0) = \frac{w_0}{z_0}, \quad g(z_0) = f^*(z_0),$$

so that, by applying Pick's Lemma to h and g,

$$\rho(f^*(0), w_0/z_0) \le \rho(0, z_0), \qquad \rho(f^*(z_0), w_0/z_0) \le \rho(0, z_0). \tag{4}$$

This shows that $f^*(0)$ and $f^*(z)$ lie in the closed disc D.

The inequality (2) (with z replaced by z_0) is an immediate consequence of (4) and the Triangle inequality, and this completes the proof of the Theorem. For a Euclidean version of Lemma 1, see [1], p. 19 and [3], p. 198.

We end with the remark that if equality holds in (2), then $f^*(0)$ and $f^*(z)$ must lie at diametrically opposite points on the boundary of D, so that equality holds in both of the inequalities in (4). We deduce that h is a Möbius map of $\mathbb{D}$ onto itself and hence $f(z) = zh(z)$, a Blaschke product of degree two.

References

1. C. Caratheodory, *Theory of functions of a complex variable*, Vol. II, Chelsea, 1960. MR **16:**346c
2. J. Dieudonne, *Recherches sur quelques problemes relatifs aux polynomes et aux fonctions bornees d'une variable complexe*, Ann. Sci. Ecole Norm. Sup. **48** (1931), 247–358.
3. P. L. Duren, *Univalent functions*, Springer-Verlag, 1983. MR **85j:**30034

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