

A RENEWAL THEOREM IN THE FINITE-MEAN CASE

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(Communicated by Stanley Sawyer)

ABSTRACT. Let $F(\cdot)$ be a c.d.f. on $(0, \infty)$ such that $\bar{F}(\cdot) \equiv 1 - F(\cdot)$ is regularly varying with exponent $-\alpha$, $1 < \alpha < 2$. Then $U(t) - \frac{t}{\mu} - \frac{1}{\mu^2} \int_0^t \int_s^\infty \bar{F}(v) dv ds = O(t^4 \bar{F}(t)^2 \bar{F}(t^2 \bar{F}(t)))$ as $t \rightarrow \infty$, where $U(t) = EN(t)$ is the renewal function associated with $F(t)$. Moreover similar estimates are given for distributions in the domain of attraction of the normal distribution and for the variance of $N(t)$. The estimates improve earlier results of Teugels and Mohan.

1. INTRODUCTION AND RESULTS

In this paper we assume that X_1, X_2, \dots is a sequence of i.i.d. real-valued positive random variables with d.f. F . Define the associated random walk by $S_0 = 0, S_n = X_1 + \dots + X_n$ for $n \geq 1$, $N(t) = \max \{n \geq 0; S_n \leq t\}$ and the renewal function U by

$$U(t) \equiv EN(t) = \sum_{n=0}^{\infty} P(S_n \leq t).$$

If F is not arithmetic, Blackwell's theorem says that

$$U(t) - U(t-h) \rightarrow \frac{h}{\mu} \text{ as } t \rightarrow \infty$$

for every $h > 0$ where $\mu = EX_1$. See Feller [5] for a proof and an extension to the arithmetic case. Extensions such as

$$U(t) - \frac{t}{\mu} \rightarrow \frac{\sigma^2 + \mu^2}{2\mu^2} \text{ as } t \rightarrow \infty$$

in case σ^2 is finite are usually proved using so-called key renewal theorems which give the asymptotic behavior of the convolution

$$(U * Q)(t) \equiv \int_{0-}^t Q(t-u)U(du)$$

as $t \rightarrow \infty$ under suitable hypothesis on $Q(\cdot)$ and $F(\cdot)$.

Received by the editors March 12, 1996 and, in revised form, June 21, 1996.

1991 *Mathematics Subject Classification*. Primary 60K05.

Key words and phrases. Renewal function, regular variation, key renewal theorem, domain of attraction.

In this paper we consider the special case where F is not arithmetic and the distribution function tail $\bar{F} \equiv 1 - F$ is regularly varying, i.e. satisfies

$$(1.1) \quad \frac{\bar{F}(tx)}{\bar{F}(t)} \rightarrow x^{-\alpha} \text{ as } t \rightarrow \infty.$$

In this case we use the notation $\bar{F}(\cdot) \in RV_{-\alpha}$. Following earlier work by Feller [4] and Smith [10, 11], Teugels [12] addressed the question of the asymptotic behavior of the convolution $U * Q(\cdot)$ under the assumption $\bar{F} \in RV_{-\alpha}$ where $0 < \alpha < 2$. In case $\frac{1}{2} < \alpha < 1$ (which implies $\mu = \infty$) an improvement of Teugels' result is given by Anderson and Athreya [1] using a result of Erickson [3]. In the case of a regularly varying tail function with $1 < \alpha < 2$ (in which case $\mu < \infty$ and $\sigma^2 = \infty$) Teugels [12] proved that

$$(1.2) \quad \tau(t) \equiv U(t) - \frac{t}{\mu} \sim \frac{t^2 \bar{F}(t)}{\mu^2(\alpha - 1)(2 - \alpha)} \text{ as } t \rightarrow \infty$$

(whence $\tau(\cdot) \in RV_{2-\alpha}$) under a supplementary condition. This condition was shown to be unnecessary in a paper by Mohan [8]. Besides this, Mohan proved that for $F \in D(\alpha)$, the domain of attraction of a stable law with exponent α where $1 < \alpha \leq 2$ (in case $\alpha = 2$ F is assumed to have infinite variance), the above function $\tau(\cdot)$ satisfies the asymptotic relation

$$(1.3) \quad \tau(t) \sim \frac{1}{\mu^2} \int_0^t \int_x^\infty \bar{F}(v) dv dx \text{ as } t \rightarrow \infty.$$

It should be observed that the above asymptotic relation holds for non-arithmetic F with finite mean μ even without the assumption (1.1). See Frenk [6], Lemma 4.1.2. Note that in case $1 < \alpha < 2$ relation (1.3) is equivalent to (1.2). In this paper we show that the assumption $F \in D(\alpha)$ with $1 < \alpha \leq 2$ permits a stronger conclusion than (1.3). In particular we have the following results.

Theorem 1.1. *Suppose $F \in D(\alpha)$ with $1 < \alpha \leq 2$ (where in case $\alpha = 2$ we assume σ^2 infinite) is not arithmetic. Suppose $Q(t) = \int_t^\infty q(s) ds < \infty$, $t \geq 0$ where $q(\cdot)$ is nonnegative and nonincreasing. Suppose $\int_0^t Q(s) ds \in RV_{-\gamma+1}$ ($0 < \gamma \leq 1$), where in case $\gamma = 1$ we assume $tQ(t) \rightarrow \infty$ as $t \rightarrow \infty$.*

$$(1.4) \quad (U * Q)(t) = \begin{cases} \frac{1}{\mu} \int_0^t Q(s) ds + O(\tau(t)Q(\tau(t))) & \text{if } 0 < \gamma < 1, \\ \frac{1}{\mu} \int_0^t Q(s) ds + o(\int_0^{\tau(t)} Q(s) ds) & \text{if } \gamma = 1. \end{cases}$$

Theorem 1.2. *Suppose $F \in D(\alpha)$ with $1 < \alpha \leq 2$ (where in case $\alpha = 2$ we assume σ^2 infinite) is not arithmetic. Then as $t \rightarrow \infty$*

$$(1.5) \quad \begin{cases} U(t) - \frac{t}{\mu} - \frac{1}{\mu^2} \int_0^t \int_s^\infty \bar{F}(v) dv ds = \\ O(\tau(\tau(t))) = O(t^4 \bar{F}(t)^2 \bar{F}(t^2 \bar{F}(t))) & \text{if } 1 < \alpha < 2, \\ o(\tau(\tau(t))), \text{ where } \tau(\cdot) \text{ satisfies (1.3)} & \text{if } \alpha = 2. \end{cases}$$

We may use the same technique as Smith [11] to find the variance of $N(t)$. Using (1.5) and the method used in Smith's paper gives the following estimate which improves the estimates given in the papers of Teugels [12] and Mohan [8]. We omit the details of the proof.

Theorem 1.3. *Under the assumptions of Theorem 1.2 we have*

$$(1.6) \quad \begin{aligned} & \text{var } N(t) - \frac{4}{\mu^3} \int_0^t \int_0^u \int_v^\infty \bar{F}(s) ds dv du + \frac{2t}{\mu^3} \int_0^t \int_v^\infty \bar{F}(s) ds dv = \\ & \begin{cases} O(t^5 \bar{F}(t)^2 \bar{F}(t^2 \bar{F}(t))) & \text{if } 1 < \alpha < 2, \\ o(t\tau(\tau(t))), \text{ where } \tau(\cdot) \text{ satisfies (1.3)} & \text{if } \alpha = 2. \end{cases} \end{aligned}$$

2. PROOFS

In the proofs below we write $\beta = 2 - \alpha$. Before giving the proofs of the results we list some well-known properties of RV functions which are used in the sequel. For a proof the reader is referred to Bingham et al. [2], Geluk and de Haan [7] or Resnick [9].

Lemma 1. *Suppose $\phi \in RV_\delta$. It follows that*

- (i) *Uniform convergence theorem for regularly varying functions. Convergence in $\phi(tx)/\phi(t) \rightarrow x^\delta (t \rightarrow \infty)$ is uniform on compact intervals of $(0, \infty)$.*
- (ii) *Karamata's theorem. There exists $t_0 > 0$ such that $\phi(t)$ is positive and locally bounded for $t > t_0$. If $\delta \geq -1$, then*

$$\lim_{t \rightarrow \infty} \frac{t\phi(t)}{\int_{t_0}^t \phi(s) ds} = \delta + 1.$$

A similar result holds for $\delta < -1$.

- (iii) *Potter's inequality. If $\varepsilon_1, \varepsilon_2 > 0$ are arbitrary, there exists $t_0 = t_0(\varepsilon_1, \varepsilon_2)$ such that for $t \geq t_0, tx \geq t_0$*

$$(1 - \varepsilon_1)x^{\delta - \varepsilon_2} < \frac{\phi(tx)}{\phi(t)} < (1 + \varepsilon_1)x^{\delta + \varepsilon_2}.$$

- (iv) *Monotone density theorem. If $\delta \geq 0$ and $\phi(t) = \int_0^t f(s) ds$ for $t \geq 0$ with f monotone, then $\lim_{t \rightarrow \infty} tf(t)/\phi(t) = \delta$. Hence in case $\delta > 0$ we have $f(\cdot) \in RV_{\delta-1}$.*

Proof of Theorem 1.1. We write

$$\int_{0-}^t Q(t-y)U(dy) = I_1 + I_2 + I_3,$$

where I_1, I_2 and I_3 are the integrals over $(0, [L(t)]), ([L(t)], [t])$ and $([t], t)$ respectively, where $[t]$ denotes the greatest integer not exceeding t . Take $L(t) \rightarrow \infty (t \rightarrow \infty)$ a slowly varying (i.e. in RV_0) function. In case $\beta = 0$ take in addition $L(t) = o(\tau(t)) (t \rightarrow \infty)$, which is possible since $\tau(t) \rightarrow \infty$ as $t \rightarrow \infty$. (Note that F has infinite variance.)

First we estimate $I_1 = \int_0^{[L(t)]} Q(t-y)U(dy)$. By monotonicity of Q we have as $t \rightarrow \infty$

$$0 \leq I_1 \leq Q(t - [L(t)])U(L(t)).$$

Since $L(t) = o(t)$ and $\tau(\cdot) \in RV_\beta, 0 \leq \beta < 1$, we have $\tau(t) = o(t - L(t))$ as $t \rightarrow \infty$; hence $Q(t - [L(t)]) \leq Q(\tau(t))$ for t sufficiently large. It follows that $I_1 = O(L(t)Q(\tau(t))) = o(\tau(t)Q(\tau(t)))$.

The second integral is estimated as follows.

$$\begin{aligned}
 I_2 &= \sum_{j=[L(t)]+1}^{[t]} \int_{j-1}^j Q(t-y)U(dy) \\
 &\leq \sum_{j=[L(t)]+1}^{[t]} Q(t-j)(U(j) - U(j-1)) \\
 (2.1) \quad &= \frac{1}{\mu} \sum_{j=[L(t)]+1}^{[t]} Q(t-j) + \sum_{j=[L(t)]+1}^{[t]} Q(t-j)(\tau(j) - \tau(j-1))
 \end{aligned}$$

and similarly

$$\begin{aligned}
 I_2 &\geq \sum_{j=[L(t)]+1}^{[t]} Q(t-j+1)(U(j) - U(j-1)) \\
 (2.2) \quad &= \frac{1}{\mu} \sum_{j=[L(t)]}^{[t]-1} Q(t-j) + \sum_{j=[L(t)]+1}^{[t]} Q(t-j+1)(\tau(j) - \tau(j-1)).
 \end{aligned}$$

Application of the Lemmas 2, 4 and 5 below shows that $I_2 = \frac{1}{\mu} \int_0^t Q(s)ds + O(\tau(t)Q(\tau(t))) + o(\int_0^{\tau(t)} Q(s)ds)$ as $t \rightarrow \infty$. From Lemma 1 (ii) and (iv) in case $0 < \gamma < 1$ it follows that $\int_0^t Q(s)ds \sim (1 - \gamma)^{-1}tQ(t)$ and $tQ(t) \rightarrow \infty$ ($t \rightarrow \infty$). In case $\gamma = 1$ we have $tQ(t) = o(\int_0^t Q(s)ds)$. The proof is complete since $I_3 = \int_{[t]}^t Q(t-y)U(dy) = O(U(t) - U([t])) = O(1)$ by Blackwell's theorem. \square

In the Lemmas 2 to 5 below the assumptions of Theorem 1.1 are supposed to be satisfied.

Lemma 2.

$$\sum_{j=[L(t)]+1}^{[t]} Q(t-j) = \int_0^t Q(s)ds + O(\tau(t)Q(\tau(t))) \quad (t \rightarrow \infty).$$

The same estimate holds for $\sum_{j=[L(t)]}^{[t]-1} Q(t-j)$.

Proof of Lemma 2. Since the second statement is equivalent to the first (note that $Q(0+) < \infty$) we only prove the first statement. Note that by the monotonicity of Q

$$\sum_{j=[L(t)]+1}^{[t]} Q(t-j) \geq \int_{t-[t]}^{t-[L(t)]} Q(s)ds = \int_0^t Q(s)ds - \int_{t-[L(t)]}^t Q(s)ds + O(1).$$

The last integral can be estimated by $0 \leq \int_{t-[L(t)]}^t Q(s)ds \leq Q(t - [L(t)])L([t])$; hence the integral is $O(\tau(t)Q(\tau(t)))$, $t \rightarrow \infty$ using the same argument as in the proof above.

Similarly we have

$$\sum_{j=[L(t)]+1}^{[t]} Q(t-j) \leq Q(0+) + \int_0^{t-[L(t)]-1} Q(s)ds = \int_0^t Q(s)ds + O(\tau(t)Q(\tau(t))).$$

\square

Lemma 3. As $t \rightarrow \infty$

$$\phi(t) \equiv \int_{[L(t)]}^{[t]-[\tau(t)]} q(t-s)\tau(s)ds = O(\tau(t)Q(\tau(t)))$$

Proof of Lemma 3. We estimate

$$\phi(t) = t \int_{\frac{[L(t)]}{t}}^{\frac{[t]-[\tau(t)]}{t}} q(t(1-u))\tau(tu)du$$

as follows. Note that $L(t) \rightarrow \infty$ and $\tau(t) \rightarrow \infty$ ($t \rightarrow \infty$); hence $\tau(\cdot)$ is positive on the specified interval of integration for t sufficiently large. From (1.3) it follows that $\tau(\cdot)$ is asymptotic to a non-decreasing function. It follows that for $\varepsilon > 0$ arbitrary there exists $t_0 = t_0(\varepsilon)$ such that $\tau(tu) \leq (1 + \varepsilon)\tau([t] - [\tau(t)])$ uniformly for $u \in [\frac{[L(t)]}{t}, \frac{[t]-[\tau(t)]}{t}]$ as $t > t_0(\varepsilon)$. Hence we obtain

$$\phi(t) \leq (1 + \varepsilon)t\tau([t] - [\tau(t)]) \int_{\frac{[L(t)]}{t}}^{\frac{[t]-[\tau(t)]}{t}} q(t(1-u))du.$$

Regular variation of the function $\tau(\cdot)$ with exponent $\beta \in [0, 1)$ implies

$$\tau([t] - [\tau(t)]) \sim \tau(t) \quad \text{as } t \rightarrow \infty;$$

hence

$$\phi(t) \leq (1 + \varepsilon)^2\tau(t) \int_{t-[\tau(t)]}^{t-[L(t)]} q(s)ds \leq (1 + \varepsilon)^2\tau(t)Q([\tau(t)])$$

for t sufficiently large. □

Lemma 4.

$$S_1(t) \equiv \sum_{j=[L(t)]+1}^{[t]-[\tau(t)]} Q(t-j)(\tau(j) - \tau(j-1)) = O(\tau(t)Q(\tau(t)))$$

as $t \rightarrow \infty$. The same estimate holds if $Q(t-j)$ is replaced with $Q(t-j+1)$.

Proof of Lemma 4. We only prove the first estimate, the second can be proved similarly. Using partial summation we have

$$\begin{aligned} |S_1(t)| &= |-\Sigma + Q(t - [t] + [\tau(t)])\tau([t] - [\tau(t)]) - Q(t - [L(t)])\tau([L(t)])| \\ (2.3) \quad &\leq \Sigma + Q(t - [t] + [\tau(t)])\tau([t] - [\tau(t)]) + Q(t - [L(t)])\tau([L(t)]), \end{aligned}$$

where

$$\Sigma = \sum_{j=[L(t)]+1}^{[t]-[\tau(t)]} (Q(t-j) - Q(t-j+1))\tau(j-1).$$

The middle term on the right-hand side in equation (2.3) is dominated by $(1 + \varepsilon)\tau(t)Q(\tau(t) - 1)$ and the last term is asymptotic to

$$\tau(L(t))Q(t - [L(t)]) = O(\tau(t)Q(\tau(t)))$$

by the same argument as in the proof of Theorem 1.1. Note that since $L(t) \rightarrow \infty$ and $\tau(t) \rightarrow \infty$ ($t \rightarrow \infty$), monotonicity of Q implies that $\Sigma \geq 0$ for t sufficiently

large. We proceed with the upper estimate. For $\varepsilon > 0$ arbitrary and t sufficiently large we have

$$\begin{aligned} \Sigma &= \sum_{j=[L(t)+1}^{[t]-[\tau(t)]} \tau(j-1) \int_{t-j}^{t-j+1} q(s)ds \\ &\leq (1 + \varepsilon) \sum_{j=[L(t)+1}^{[t]-[\tau(t)]} \int_{t-j}^{t-j+1} \tau(t-s)q(s)ds \\ &= (1 + \varepsilon) \sum_{j=[L(t)+1}^{[t]-[\tau(t)]} \int_{j-1}^j \tau(s)q(t-s)ds \\ &= (1 + \varepsilon) \int_{[L(t)]}^{[t]-[\tau(t)]} \tau(s)q(t-s)ds. \end{aligned}$$

Note that the above inequality follows from (1.3) as in the proof of Lemma 3 above. Application of Lemma 3 completes the proof. \square

Lemma 5.

$$S_2(t) \equiv \sum_{j=[t]-[\tau(t)+1}^{[t]} Q(t-j+1)(\tau(j) - \tau(j-1)) = o\left(\int_0^{\tau(t)} Q(s)ds\right)$$

as $t \rightarrow \infty$. The same estimate holds if $Q(t-j+1)$ is replaced with $Q(t-j)$.

Proof of Lemma 5. Since $\tau(t) - \tau(t-1) = U(t) - U(t-1) - \frac{1}{\mu} \rightarrow 0$ as $t \rightarrow \infty$ by Blackwell’s theorem, it follows that for $\varepsilon > 0$ and $t > t(\varepsilon)$

$$\begin{aligned} |S_2(t)| &\leq \varepsilon \sum_{j=[t]-[\tau(t)+1}^{[t]} Q(t-j+1) \\ &\leq \varepsilon\{Q(0+) + \int_0^{t-[\tau(t)]} Q(s)ds\}. \end{aligned}$$

Since $\int_0^t Q(s)ds$ is regularly varying, the integral on the right-hand side is asymptotic to $\int_0^{\tau(t)} Q(s)ds$. The statement of the lemma is proved since $\varepsilon > 0$ is arbitrary and $\int_0^{\tau(t)} Q(s)ds \geq \tau(t)Q(\tau(t)) \rightarrow \infty$ as $t \rightarrow \infty$. The proof of the second statement is similar. \square

Proof of Theorem 1.2. Integrating the renewal equation $\int_{0-}^s \bar{F}(s-y)U(dy) = 1$ over the interval $[0, t]$ ($t > 0$) gives $\int_{0-}^t \int_0^{t-y} \bar{F}(u)duU(dy) = t$. It follows that

$$U(t) - \frac{t}{\mu} = \int_{0-}^t Q(t-y)U(dy)$$

where $Q(t) = \frac{1}{\mu} \int_t^\infty \bar{F}(s)ds$.

Since the asymptotic behavior of τ is given by (1.3), application of Theorem 1.1 completes the proof. (Note that $t^2\bar{F}(t) \geq \int_{0-}^t x^2F(dx) \rightarrow \infty$; hence $tQ(t) \rightarrow \infty$ as $t \rightarrow \infty$ in case $\alpha = 2$.) \square

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