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MEHLER-FOCK TRANSFORMS OF GENERALIZED FUNCTIONS VIA THE METHOD OF ADJOINTS

BENITO J. GONZÁLEZ AND EMILIO R. NEGRIN

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Abstract. In this paper we analyse the Mehler-Fock transform of generalized functions via the method of adjoints. For a distribution of compact support, we prove that its Mehler-Fock transform agrees with its transform via the kernel method. A Paley-Wiener type theorem is established.

1. Introduction

The Mehler-Fock transform of a distribution f of compact support on I (from now on $I = (0, \infty)$) was defined by the kernel method in [10] as:

$$F(\tau) = \langle f(x), P_{-\frac{1}{2}+i\tau}(\cosh x)\rangle, \quad \tau > 0, \tag{1.1}$$

where, as it is usual, $P_{-\frac{1}{2}+i\tau}$ is the Legendre function of first kind and order zero.

Related studies on Mehler-Fock transforms of generalized functions have been carried out in [3], [4], [5], [8], amongst others.

In this paper, we study the Mehler-Fock transform by using the method of adjoints. For this purpose, it will be required to introduce two Fréchet spaces V and W satisfying the condition that the linear operator $\mathcal{L} : W \longrightarrow V$ given by

$$(\mathcal{L}\psi)(x) = \int_0^\infty P_{-\frac{1}{2}+i\tau}(\cosh x)\psi(\tau)d\tau, \quad x > 0, \tag{1.2}$$

be continuous.

Specifically, we denote by V the vector space consisting of all complex-valued functions ϕ which possess derivatives of all orders on I and such that, for all $k \in \mathbb{N} \cup \{0\}$,

$$\gamma_k(\phi) = \sup_{x \in I} \left|(\sinh x) A_x^k \phi(x)\right| < \infty,$$

where A_x is the differential operator

$$A_x \equiv D_x^2 + (\coth x)D_x \equiv (\sinh x)^{-1} D_x (\sinh x) D_x,$$

and D_x denotes derivative with respect to the x-variable.

The topology of V is that generated by the family of seminorms $\{\gamma_k\}_{k \in \mathbb{N}\cup\{0\}}$.

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Moreover, W is the vector space of all complex-valued even functions ψ on $\mathbb{R}$ such that

$$\tau^{-1}\left(\frac{1}{4}+\tau^2\right)^r \cosh(\pi\tau)\psi(\tau) \in L^1(\mathbb{R})$$

and

$$\rho_r(\psi) = \sup_{x\in I}\left| x^{\frac{3}{2}} e^x \left(\mathcal{F}\left[\tau^{-1}\left(\frac{1}{4}+\tau^2\right)^r \cosh(\pi\tau)\psi(\tau)\right]\right)(x)\right| < \infty,$$

for all $r \in \mathbb{N}\cup\{0\}$, and $\mathcal{F}$ denotes Fourier transform with respect to the τ-variable. The space W is equipped with the topology generated by the family of seminorms $\{\rho_r\}_{r\in\mathbb{N}\cup\{0\}}$.

As usual V' and W' denote the dual spaces of V and W, respectively.

The Mehler-Fock transform of the generalized function $f \in V'$ is given by $\mathcal{L}'f$, where $\mathcal{L}'$ is the adjoint of $\mathcal{L}$.

Here, we establish the link between these two methods. It will be proved that both definitions agree for distributions of compact support on I.

We also establish a Paley-Wiener type theorem. It is proved that every smooth function F such that $|F(\tau)| \le C\cdot(1+\tau^{2r})$, for all $\tau > 0$, some nonnegative integer r and some constant C, is the Mehler-Fock transform of a generalized function $f \in V'$.

In this article we extend this analysis to the so-called Mehler-Fock transform of order $n \in \mathbb{N}$.

The corresponding results for the Kontorovich-Lebedev transform were carried out in [7].

2. The operator $\mathcal{L}$ and its adjoint

The next result is the key to analysing the Mehler-Fock transform of generalized functions via the method of adjoints.

Theorem 2.1. *The operator $\mathcal{L}$, given by* (1.2)*, is a continuous linear mapping from W in V.*

Proof. First observe that for all $\tau > 0$,

$$A_x P_{-\frac{1}{2}+i\tau}(\cosh x) = -\left(\frac{1}{4}+\tau^2\right) P_{-\frac{1}{2}+i\tau}(\cosh x).$$

So, for any $\psi \in W$ and all $k \in \mathbb{N}\cup\{0\}$, we have

$$A_x^k\left(\mathcal{L}\psi\right)(x) = (-1)^k \int_0^\infty \left(\frac{1}{4}+\tau^2\right)^k P_{-\frac{1}{2}+i\tau}(\cosh x)\psi(\tau)d\tau. \tag{2.1}$$

Using the integral representation (cf. [1, (11), p. 156]), which is valid for all $x > 0$ and all $\tau > 0$,

$$P_{-\frac{1}{2}+i\tau}(\cosh x) = \frac{\sqrt{2}}{\pi}\cosh(\pi\tau)\int_0^\infty (\cosh x + \cosh\xi)^{-\frac{1}{2}}\cos(\tau\xi)d\xi \tag{2.2}$$

and integrating (2.2) by parts, one obtains

$$P_{-\frac{1}{2}+i\tau}(\cosh x) = \frac{\cosh(\pi\tau)}{\sqrt{2}\pi\tau}\int_0^\infty (\cosh x + \cosh\xi)^{-\frac{3}{2}}(\sin(\tau\xi))(\sinh\xi)d\xi.$$

Substituting this expression in (2.1) and reversing the order of integration, it follows that

$$A_x^k(\mathcal{L}\psi)(x) = \frac{(-1)^k}{\sqrt{2}\pi}\int_0^\infty (\cosh x+\cosh\xi)^{-\frac{3}{2}}(\sinh\xi) \times\left(\int_0^\infty \tau^{-1}\cosh(\pi\tau)\left(\frac{1}{4}+\tau^2\right)^k\psi(\tau)\sin(\tau\xi)d\tau\right)d\xi. \tag{2.3}$$

Note that the integral into the brackets is the Fourier sine transform of an odd function of the τ-variable, and consequently it is the Fourier transform of this function.

Therefore,

$$A_x^k(\mathcal{L}\psi)(x) = \frac{(-1)^k}{\sqrt{2}\pi}\int_0^\infty (\cosh x+\cosh\xi)^{-\frac{3}{2}}(\sinh\xi) \times\left(\mathcal{F}\left[\tau^{-1}\cosh(\pi\tau)\left(\frac{1}{4}+\tau^2\right)^k\psi(\tau)\right]\right)(\xi)\,d\xi. \tag{2.4}$$

Thus, for all $x>0$ we have

$$\left|(\sinh x)A_x^k(\mathcal{L}\psi)(x)\right| = \left|\frac{\sinh x}{\sqrt{2}\pi(\cosh x)^{\frac{3}{2}}}\int_0^\infty\left(1+\frac{\cosh\xi}{\cosh x}\right)^{-\frac{3}{2}}(\sinh\xi) \times\left(\mathcal{F}\left[\tau^{-1}\cosh(\pi\tau)\left(\frac{1}{4}+\tau^2\right)^k\psi(\tau)\right]\right)(\xi)\,d\xi\right|$$

which is less than or equal to

$$\frac{1}{\sqrt{2}\pi}\int_0^\infty e^{-\xi}\xi^{-\frac{3}{2}}(\sinh\xi)\left|e^{\xi}\xi^{\frac{3}{2}}\left(\mathcal{F}\left[\tau^{-1}\cosh(\pi\tau)\left(\frac{1}{4}+\tau^2\right)^k\psi(\tau)\right]\right)(\xi)\right|d\xi \leq M\rho_k(\psi),$$

where M is a suitable real constant. Thus, for all $k\in\mathbb{N}\cup\{0\}$,

$$\gamma_k(\mathcal{L}\psi)\leq M\rho_k(\psi).$$

This concludes the continuity of $\mathcal{L}$. ■

Now, the adjoint of the operator $\mathcal{L}$, which is given by

$$\langle\mathcal{L}'f,\psi\rangle = \langle f,\mathcal{L}\psi\rangle,\quad f\in V',\quad \psi\in W,$$

is a continuous linear mapping from V' into W' when we provide V' and W' with their weak*-topologies. The Mehler-Fock transform of the generalized function f is the member of W' given by $\mathcal{L}'f$.

3. The link between both methods

The next two lemmas allow us to establish the link between the two definitions of Mehler-Fock transforms of generalized functions.

For it, set

$$\psi_\phi(\tau) = \tau\tanh(\pi\tau)\int_0^\infty(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)\phi(x)dx, \tag{3.1}$$

for any $\phi\in V$ and all $\tau>0$.

Lemma 3.1. *For all $\phi \in V$ the function ψ_ϕ given by* (3.1) *satisfies*
i) $\psi_\phi(\tau) = O(\tau^2)$, *as* $\tau \to 0$;
ii) *for all* $p \in \mathbb{N}$, $\psi_\phi(\tau) = O(\tau^{-p})$, *as* $\tau \to \infty$.

Proof. i) Taking into account that $\tanh(\pi\tau) = O(\tau)$ for $\tau \to 0$, it follows that for τ small enough,

$$|\psi_\phi(\tau)| \le C_1\tau^2 \int_0^\infty \left|(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)\phi(x)\right| dx \le C_2\tau^2,$$

for some suitable real constants C_1 and C_2.

ii) If A'_x denotes the adjoint of A_x, namely, $A'_x = D_x(\sinh x)D_x(\sinh x)^{-1}$, one has

$$A'_x\left[(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)\right] = -\left(\frac{1}{4}+\tau^2\right)(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x),$$

and so, for all $k \in \mathbb{N}\cup\{0\}$,

$$(-1)^k\left(\frac{1}{4}+\tau^2\right)^k\psi_\phi(\tau) = \tau\tanh(\pi\tau)\int_0^\infty (A'_x)^k\left[(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)\right]\phi(x)dx. \tag{3.2}$$

Note that the integral in the right-hand side of (3.2) is equal to

$$\int_0^\infty (\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)A_x^k\phi(x)dx.$$

In fact, integrating by parts,

$$\begin{aligned}\int_0^\infty & A'_x\left[(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)\right]\phi(x)dx\\ &= (\sinh x)\left(D_xP_{-\frac{1}{2}+i\tau}(\cosh x)\right)\phi(x)\Big|_0^\infty\\ &\quad - \int_0^\infty(\sinh x)\left(D_xP_{-\frac{1}{2}+i\tau}(\cosh x)\right)(D_x\phi(x))dx\\ &= (\sinh x)\left(D_xP_{-\frac{1}{2}+i\tau}(\cosh x)\right)\phi(x)\Big|_0^\infty\\ &\quad - (\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)(D_x\phi(x))\Big|_0^\infty\\ &\quad + \int_0^\infty(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)A_x\phi(x)dx.\end{aligned}$$

Observe that the remainder terms of the above integration by parts tend to zero as $x \to 0$ and $x \to \infty$. Indeed, since $\phi \in V$, it follows that $(\sinh x)\phi(x)$ is bounded on I, and

$$D_xP_{-\frac{1}{2}+i\tau}(\cosh x) \longrightarrow 0, \text{ as } x \to 0 \text{ and } x \to \infty.$$

Hence,

$$(\sinh x)(D_xP_{-\frac{1}{2}+i\tau}(\cosh x))\phi(x) \longrightarrow 0, \text{ as } x \to 0 \text{ and } x \to \infty.$$

Also, since for all $\phi \in V$, the function $(\sinh x)D_x\phi(x)$ is bounded on I, and taking into account that $P_{-\frac{1}{2}+i\tau}(\cosh x) = O(e^{-\frac{x}{2}})$ as $x \to \infty$, it follows that

$$(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)D_x\phi(x) \longrightarrow 0, \text{ as } x \to \infty.$$

On the other hand, for any $\phi \in V$ one has

$$(\sinh x) D_x \phi(x) = O(x), \quad \text{as } x \to 0.$$

Thus,

$$(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) D_x \phi(x) \longrightarrow 0, \quad \text{as } x \to 0.$$

This proves that

$$-\left(\frac{1}{4}+\tau^2\right)\psi_\phi(\tau) = \tau \tanh(\pi\tau) \int_0^\infty (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) A_x \phi(x) dx.$$

Now, for all $k \in \mathbb{N}$, $k > 1$, one obtains

$$\int_0^\infty \left(A'_x\right)^k \left[(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x)\right] \phi(x) dx$$

$$= (-1)^{k-1}\left(\frac{1}{4}+\tau^2\right)^{k-1} \int_0^\infty A'_x \left[(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x)\right] \phi(x) dx$$

$$= (-1)^{k-1}\left(\frac{1}{4}+\tau^2\right)^{k-1} \int_0^\infty (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) A_x \phi(x) dx.$$

Iterating this process one arrives at

$$(-1)^k\left(\frac{1}{4}+\tau^2\right)^k \psi_\phi(\tau) = \tau \tanh(\pi\tau) \int_0^\infty (\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) A_x^k \phi(x) dx.$$

Taking into account that $\phi \in V$ and the asymptotic behavior of $P_{-\frac{1}{2}+i\tau}(\cosh x)$ as $x \to 0$ and $x \to \infty$, one concludes that

$$\left(\frac{1}{4}+\tau^2\right)^k |\psi_\phi(\tau)| \leq M\tau, \quad \forall \tau > 0,$$

for some suitable nonnegative constant M. Thus, the lemma holds. ■

Lemma 3.2. *For all $\psi \in W$, there exists $\phi \in V$ such that $\psi = \psi_\phi$, where ψ_ϕ is given by* (3.1).

Proof. Let $\psi \in W$ and set $\phi = \mathcal{L}\psi$. Note that, for $x > 0$, one has

$$(\mathcal{L}\psi_\phi)(x) = \int_0^\infty P_{-\frac{1}{2}+i\tau}(\cosh x)\psi_\phi(\tau) d\tau$$

$$= \int_0^\infty P_{-\frac{1}{2}+i\tau}(\cosh x) \tau \tanh(\pi\tau) \int_0^\infty (\sinh t) P_{-\frac{1}{2}+i\tau}(\cosh t)\phi(t) dt \, d\tau,$$

and from the inversion formula of the Mehler-Fock transform, the above expression is equal to $\phi(x)$, which holds whenever $\phi(x)e^{\frac{x}{2}} \in L^1(0,\infty)$ (cf. [2]). Since $\phi \in V$, this condition is satisfied.

Thus,

$$\mathcal{L}\psi_\phi = \phi = \mathcal{L}\psi.$$

Now, we will establish that $\mathcal{L}$ is one to one. For it, assume that $\mathcal{L}\psi = 0$. Taking $k = 0$ in (2.4) one has for all $x > 0$,

$$(\mathcal{L}\psi)(x) = \frac{1}{\sqrt{2}\pi} \int_0^\infty (\cosh x + \cosh \xi)^{-\frac{3}{2}} (\sinh \xi) \left(\mathcal{F}\left[\tau^{-1}\cosh(\pi\tau)\psi(\tau)\right]\right)(\xi) \, d\xi = 0.$$

The change of variables $t+1=\cosh\xi$ yields

$$0=(\mathcal{L}\psi)(x)=\frac{1}{\sqrt{2}\pi}\int_0^\infty (\cosh x+1+t)^{-\frac{3}{2}}\Psi(t)dt,\ \ \forall x>0,$$

where

$$\Psi(t)=\left(\mathcal{F}\left[\tau^{-1}\cosh(\pi\tau)\psi(\tau)\right]\right)(\log\,(t+1+\sqrt{t^2+2t})).$$

Finally, since

$$\int_0^\infty (\cosh x+1+t)^{-\frac{3}{2}}\Psi(t)dt$$

is the generalized Stieltjes transform of the function Ψ, from its inversion formula (cf. [9, p. 180]) it follows that $\Psi\equiv 0$. Therefore,

$$\left(\mathcal{F}\left[\tau^{-1}\cosh(\pi\tau)\psi(\tau)\right]\right)(\log\,(t+1+\sqrt{t^2+2t}))=0,\ \ \forall t>0,$$

whence $\psi\equiv 0$. ■

Theorem 3.1. *Let $f\in\mathcal{E}'(I)$. Then, for any $\psi\in W$, one has*

$$\langle\mathcal{L}'f,\psi\rangle=\int_0^\infty F(\tau)\psi(\tau)d\tau, \tag{3.3}$$

where the function $F(\tau)$ is given by (1.1).

Proof. First, observe that $\mathcal{D}(I)\subset V\subset\mathcal{E}(I)$ and the topology of V is stronger than the one induced on it by $\mathcal{E}(I)$. Thus, we may regard $\mathcal{E}'(I)$ as a vector subspace of V'.

By [6, Proposition 2, p. 97], the function $F(\tau)$ satisfies

i) For $\tau\to 0$,

$$F(\tau)=O(1), \tag{3.4}$$

ii) there exists $r\in\mathbb{N}\cup\{0\}$ such that for $\tau\to\infty$,

$$F(\tau)=O\left(\tau^{2r}\right). \tag{3.5}$$

Using Lemma 3.1 and Lemma 3.2 it follows that for any $\psi\in W$ one has

i) $\psi(\tau)=O(\tau^2)$, as $\tau\to 0$;

ii) for all $p\in\mathbb{N}$, $\psi(\tau)=O(\tau^{-p})$, as $\tau\to\infty$.

Thus, the integral of the right-hand side of (3.3) makes sense.

Clearly, if $f\in\mathcal{D}(I)$, the result of this theorem holds. In fact, assume that f has its support contained in the closed interval $[a,b]\subset I$; then, using Fubini's theorem, one has

$$\langle\mathcal{L}'f,\psi\rangle=\langle f,\mathcal{L}\psi\rangle$$

$$=\int_a^b f(x)(\mathcal{L}\psi)(x)dx=\int_a^b f(x)\int_0^\infty P_{-\frac{1}{2}+i\tau}(\cosh x)\psi(\tau)d\tau\ dx$$

$$=\int_0^\infty \psi(\tau)\int_a^b f(x)P_{-\frac{1}{2}+i\tau}(\cosh x)dx\ d\tau=\int_0^\infty F(\tau)\psi(\tau)d\tau.$$

So, the result holds for $f\in\mathcal{D}(I)$.

Now, for $f\in\mathcal{E}'(I)$ and using [11, Theorem 28.2(i), p. 301], there exists a sequence of functions $\{f_m\}_{m\in\mathbb{N}}$ belonging to $\mathcal{D}(I)$ which converges to f in $\mathcal{E}'(I)$, and hence, in V'. Furthermore, from Theorem 2.1 it follows that $\mathcal{L}'$ is a continuous mapping from V' into W'. Thus $\{\mathcal{L}' f_m\}_{m\in\mathbb{N}}$ converges to $\mathcal{L}' f$ in W' as $m \to \infty$.

Since, for all $\psi \in W$,

$$\langle \mathcal{L}' f_m, \psi\rangle = \int_0^\infty F_m(\tau)\psi(\tau)d\tau$$

where

$$F_m(\tau) = \langle f_m(x), P_{-\frac{1}{2}+i\tau}(\cosh x)\rangle,$$

it follows that

$$\langle \mathcal{L}' f, \psi\rangle = \lim_{m\to\infty} \int_0^\infty F_m(\tau)\psi(\tau)d\tau. \tag{3.6}$$

Clearly, from the asymptotic growth of F_m and ψ, the limit in (3.6) goes into the integral. Also, noting that

$$\begin{aligned}\lim_{m\to\infty} F_m(\tau) &= \lim_{m\to\infty} \langle f_m(x), P_{-\frac{1}{2}+i\tau}(\cosh x)\rangle \\ &= \langle f(x), P_{-\frac{1}{2}+i\tau}(\cosh x)\rangle = F(\tau),\end{aligned}$$

the result holds. ■

4. A Paley-Wiener type theorem

The following theorem proves that any smooth function which satisfies conditions of type (3.4) and (3.5) is the Mehler-Fock transformation of an element of V'.

Theorem 4.1. *Let $F(\tau)$ be a smooth function on I such that there exist a nonnegative integer r and a constant C with $|F(\tau)| \le C\cdot\left(1+\tau^{2r}\right)$ for all $\tau > 0$. Then there exists a $f \in V'$ such that*

$$\langle \mathcal{L}' f, \psi\rangle = \int_0^{+\infty} F(\tau)\psi(\tau)d\tau, \quad \forall \psi \in W.$$

Moreover, for all $\phi \in V$,

$$\langle f, \phi\rangle = \lim_{T\to\infty} \langle \int_0^T \tau\tanh(\pi\tau)(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh\tau)F(\tau)d\tau, \phi(x)\rangle.$$

Proof. Let $G(\tau)$ be a smooth function on $(0,\infty)$ such that

$$|G(\tau)| \le M_1\cdot\left(1+\tau^{-\alpha}\right), \tag{4.1}$$

where $\alpha > 2$ and M_1 is a real constant. Let us define, for any $x > 0$,

$$g(x) = \int_0^\infty \tau\tanh(\pi\tau)(\sinh x)P_{-\frac{1}{2}+i\tau}(\cosh x)G(\tau)d\tau. \tag{4.2}$$

Note that for $\tau \to 0$,

$$\tau\tanh(\pi\tau) = O(\tau^2),$$

and for $\tau \to \infty$,

$$\tau\tanh(\pi\tau) = O(\tau).$$

On the other hand,

$$\left|P_{-\frac{1}{2}+i\tau}(\cosh x)\right| \le \left|P_{-\frac{1}{2}}(\cosh x)\right|, \quad \forall x > 0, \ \ \forall \tau > 0.$$

So, (4.2) exists and it is a locally integrable function on I. Moreover,

$$\int_0^\infty \left|(\sinh x)^{-1} g(x)\right| dx = \int_0^\infty \left|\int_0^\infty \tau \tanh(\pi\tau) P_{-\frac{1}{2}+i\tau}(\cosh x) G(\tau) d\tau\right| dx$$

$$\leq \int_0^\infty \left|P_{-\frac{1}{2}}(\cosh x)\right| \int_0^\infty \tau |\tanh(\pi\tau)||G(\tau)| d\tau \; dx$$

$$\leq M_2 \cdot \int_0^\infty \left|P_{-\frac{1}{2}}(\cosh x)\right| dx < \infty,$$

where M_2 is a real constant.

Therefore, the function g yields to a regular member in V', denoted by T_g, which is given by

$$\langle T_g, \phi\rangle = \int_0^\infty g(x)\phi(x) dx, \quad \forall \phi \in V.$$

Also for any $\psi \in W$ denoting $\phi = \mathcal{L}\psi$, and using Lemma 3.2, one obtains

$$\langle \mathcal{L}' T_g, \psi\rangle = \langle T_g, \mathcal{L}\psi\rangle = \int_0^\infty g(x)\phi(x) dx$$

$$= \int_0^\infty \phi(x) \int_0^\infty \tau \tanh(\pi\tau)(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) G(\tau) d\tau \; dx,$$

which, by Fubini's theorem, is equal to

$$\int_0^\infty G(\tau) \int_0^\infty \tau \tanh(\pi\tau)(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x)\phi(x) dx \; d\tau = \int_0^\infty G(\tau)\psi(\tau) d\tau.$$

Thus, for all $\psi \in W$,

$$\langle \mathcal{L}' T_g, \psi\rangle = \int_0^\infty G(\tau)\psi(\tau) d\tau. \tag{4.3}$$

Now, for all $\tau > 0$, set

$$G(\tau) = \frac{F(\tau)}{\left(\frac{1}{4} + \tau^2\right)^{r+p}}.$$

Clearly, for $p > \alpha/2$, this function G satisfies condition (4.1). So, there exists a function g given by (4.2) such that the regular member T_g in V' satisfies (4.3). Set $f = (-A'_x)^{r+p} T_g$. Note that, for all $\phi \in V$,

$$\langle f, \phi\rangle = \langle (-A'_x)^{r+p} T_g, \phi\rangle = \int_0^\infty g(x)(-A_x)^{r+p}\phi(x) dx.$$

Thus, for all $\psi \in W$, one obtains

$$\langle \mathcal{L}' f, \psi\rangle = \langle T_g, (-A_x)^{r+p}(\mathcal{L}\psi)\rangle = \left\langle T_g, \mathcal{L}\left[\left(\frac{1}{4} + \tau^2\right)^{r+p} \psi(\tau)\right]\right\rangle$$

$$= \int_0^\infty G(\tau)\left(\frac{1}{4} + \tau^2\right)^{r+p} \psi(\tau) d\tau = \int_0^\infty F(\tau)\psi(\tau) d\tau.$$

On the other hand, for any $\phi \in V$,

$$\langle f, \phi\rangle = \langle \int_0^\infty \tau \tanh(\pi\tau)(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) G(\tau) d\tau, (-A_x)^{r+p}\phi(x)\rangle$$

$$= \lim_{T\to\infty} \langle \int_0^T \tau \tanh(\pi\tau)(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) G(\tau) d\tau, (-A_x)^{r+p}\phi(x)\rangle$$

$$= \lim_{T\to\infty} \langle \int_0^T \tau \tanh(\pi\tau)(-A_x')^{r+p} \left[(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x)\right] G(\tau) d\tau, \phi(x)\rangle$$

$$= \lim_{T\to\infty} \langle \int_0^T \tau \tanh(\pi\tau)(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) \left(\frac{1}{4}+\tau^2\right)^{r+p} G(\tau) d\tau, \phi(x)\rangle$$

$$= \lim_{T\to\infty} \langle \int_0^T \tau \tanh(\pi\tau)(\sinh x) P_{-\frac{1}{2}+i\tau}(\cosh x) F(\tau) d\tau, \phi(x)\rangle.$$

■

5. Mehler-Fock transforms of order n

Starting from the integral representation [1, (11), p. 156], which is valid for all $n \in \mathbb{N} \cup \{0\}$, all $\tau > 0$ and all $x > 0$,

$$P^n_{-\frac{1}{2}+i\tau}(\cosh x) = \sqrt{\frac{2}{\pi}} \frac{\Gamma(n+\frac{1}{2})(\sinh x)^n}{\Gamma(n+\frac{1}{2}+i\tau)\Gamma(n+\frac{1}{2}-i\tau)} \int_0^\infty (\cosh x + \cosh\xi)^{-n-\frac{1}{2}} \cos(\tau\xi) d\xi,$$

the results of the previous sections are readily extended to the so-called Mehler-Fock transform of order $n \in \mathbb{N}$, the kernel of which is given by $P^n_{-\frac{1}{2}+i\tau}(\cosh x)$, the associated Legendre function of first kind and order n.

Specifically, one introduces the corresponding spaces V and W; namely, V is the vector space consisting of all complex-valued functions ϕ which possess derivatives of all orders on I and such that, for all $k \in \mathbb{N} \cup \{0\}$,

$$\gamma_k(\phi) = \sup_{x\in I} \left|(\sinh x) A_x^k \phi(x)\right| < \infty,$$

where A_x is the differential operator

$$A_x \equiv (\sinh x)^{-n-1} D_x (\sinh x)^{2n+1} D_x (\sinh x)^{-n},$$

and W is the vector space of all complex-valued even functions ψ on $\mathbb{R}$ such that

$$\frac{\left(\left(n+\frac{1}{2}\right)^2+\tau^2\right)^r \psi(\tau)}{\tau\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right)} \in L^1(\mathbb{R})$$

and

$$\rho_r(\psi) = \sup_{x\in I} \left| x^{\frac{3}{2}} e^x \left(\mathcal{F}\left[\frac{\left(\left(n+\frac{1}{2}\right)^2+\tau^2\right)^r \psi(\tau)}{\tau\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right)}\right]\right)(x)\right| < \infty,$$

for all $r \in \mathbb{N} \cup \{0\}$, and $\mathcal{F}$ denotes Fourier transform with respect to the τ-variable.

The operator $\mathcal{L} : W \to V$, given by

$$(\mathcal{L}\psi)(x) = \int_0^\infty P^n_{-\frac{1}{2}+i\tau}(\cosh x)\psi(\tau) d\tau,$$

becomes a continuous linear mapping from W in V.

Also, for any $\phi \in V$ and all $\tau > 0$ set

$$\psi_\phi(\tau) = \frac{1}{\pi}\tau\sinh(\pi\tau)\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right) \times \int_0^\infty (\sinh x) P^n_{-\frac{1}{2}+i\tau}(\cosh x)\phi(x)dx.$$

It is obtained that
i) $\psi_\phi(\tau) = O(\tau^{2n+2})$, as $\tau \to 0$;
ii) for all $p \in \mathbb{N}$, $\psi_\phi(\tau) = O(\tau^{-p})$, as $\tau \to \infty$.
Also, for any $\psi \in W$ there exists $\phi \in V$ such that $\psi = \psi_\phi$.
Now, for any $f \in \mathcal{E}'(I)$, the function

$$F(\tau) = \langle f(x), P^n_{-\frac{1}{2}+i\tau}(\cosh x)\rangle, \quad \tau > 0,$$

satisfies
i) $F(\tau) = O(1)$, as $\tau \to 0$;
ii) there exists $r \in \mathbb{N}\cup\{0\}$ such that $F(\tau) = O(\tau^{2r})$, as $\tau \to \infty$.

Thus the corresponding result of Theorem 3.1 holds in this setting; namely, for all $f \in \mathcal{E}'(I)$,

$$\langle \mathcal{L}'f, \psi\rangle = \int_0^\infty F(\tau)\psi(\tau)d\tau.$$

Also the corresponding Paley-Wiener type theorem establishes that

Theorem 5.1. *Let $F(\tau)$ be a smooth function on I such that there exist a nonnegative integer r and a constant C with $|F(\tau)| \le C \cdot \left(1+\tau^{2r}\right)$ for all $\tau > 0$. Then there exists an $f \in V'$ such that*

$$\langle \mathcal{L}'f, \psi\rangle = \int_0^{+\infty} F(\tau)\psi(\tau)d\tau, \quad \forall \psi \in W.$$

Moreover, for all $\phi \in V$,

$$\langle f, \phi\rangle = \lim_{T\to\infty}\langle \int_0^T S(\tau)G(\tau,x)F(\tau)d\tau, \phi(x)\rangle,$$

where

$$S(\tau) = \frac{1}{\pi}\tau\sinh(\pi\tau)\Gamma\left(n+\frac{1}{2}+i\tau\right)\Gamma\left(n+\frac{1}{2}-i\tau\right)$$

and

$$G(\tau,x) = (\sinh x)\cdot P^{-n}_{-\frac{1}{2}+i\tau}(\cosh x).$$

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Departamento de Análisis Matemático, Universidad de La Laguna, 38271 Canary Islands, Spain

E-mail address: `bjglez@ull.es`
E-mail address: `enegrin@ull.es`