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STANDARD SYSTEMS FOR SEMIFINITE O*-ALGEBRAS

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ABSTRACT. We shall continue the study of standard systems which make it possible to develop the Tomita-Takesaki theory in O*-algebras. The main purpose of this paper is to give the necessary and sufficient conditions for which a standard system $(\mathcal{M}, \lambda, \lambda')$ of an O*-algebra $\mathcal{M}$, a generalized vector λ and the commutant λ' is unitarily equivalent to a standard system $\left(\mathcal{N}, K'\mu, (K'\mu)'\right)$ constructed by a standard tracial generalized vector μ for an O*-algebra $\mathcal{N}$ and a non-singular positive self-adjoint operator K' affiliated with the commutant $\mathcal{N}'_{\mathrm{w}}$ of $\mathcal{N}$.

1. Introduction

In order to develop the Tomita-Takesaki theory in O*-algebras, we have defined and studied the notion of standard system $(\mathcal{M}, \lambda, \lambda')$ of an O*-algebra $\mathcal{M}$, a generalized vector λ and the commutant λ' in [1, 6, 7] . Here we shall continue this study and study the structure of standard systems for semifinite O*-algebras. We first treat tracial generalized vectors μ for an O*-algebra $\mathcal{N}$ and give the necessary and sufficient conditions for which μ is standard. We next construct a generalized vector $K'\mu$ by a pair (K', μ) of a standard tracial generalized vector μ for $\mathcal{N}$ and a non-singular positive self-adjoint operator K' affiliated with the commutant $\mathcal{N}'_{\mathrm{w}}$ of $\mathcal{N}$, and investigate when $(\mathcal{N}, K'\mu, (K'\mu)')$ is a standard system. Further, we consider this converse. And we give the necessary and sufficient conditions for which a standard system $(\mathcal{M}, \lambda, \lambda')$ is unitarily equivalent to such a standard system $\left(\mathcal{N}, K'\mu, (K'\mu)'\right)$

2. Standard tracial generalized vectors

We begin with the definitions and the basic properties of standard generalized vectors for O*-algebras [7] . Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$. We denote by $\mathcal{L}^\dagger(\mathcal{D})$ the set of all linear operators X from $\mathcal{D}$ to $\mathcal{D}$ such that $\mathcal{D} \subset \mathcal{D}(X^*)$ and $X^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra equipped with the usual operators $X+Y, \alpha X$ and the involution $X^\dagger \equiv X^* \lceil \mathcal{D}$. A $*$-subalgebra $\mathcal{M}$ of $\mathcal{L}^\dagger(\mathcal{D})$ is said to be an *O*-algebra* on $\mathcal{D}$ in $\mathcal{H}$. An O*-algebra $\mathcal{M}$ on $\mathcal{D}$ in $\mathcal{H}$ is said to be *closed* (resp. *self-adjoint*) if

$$\mathcal{D} = \bigcap_{X \in \mathcal{M}} \mathcal{D}\left(\overline{X}\right) \qquad \left(\text{resp. } \mathcal{D} = \bigcap_{X \in \mathcal{M}} \mathcal{D}(X^*)\right),$$

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and $\mathcal{M}$ is said to be *integrable* if $X^{\dagger *} = \overline{X}$ for each $X \in \mathcal{M}$. It is clear that if $\mathcal{M}$ is integrable, then it is self-adjoint, and if $\mathcal{M}$ is self-adjoint, then it is closed. We define the *weak commutant* $\mathcal{M}'_{\mathrm{w}}$ of an O*-algebra $\mathcal{M}$ by

$$\mathcal{M}'_{\mathrm{w}} = \{C \in \mathcal{B}(\mathcal{H}) \ ; \ (CX\xi|\eta) = (C\xi|X^{\dagger}\eta) \text{ for each } X \in \mathcal{M} \text{ and } \xi, \eta \in \mathcal{D}\},$$

where $\mathcal{B}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. It is known that if $\mathcal{M}$ is self-adjoint, then $\mathcal{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$; and if $\mathcal{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$, then $\mathcal{M}'_{\mathrm{w}}$ is a von Neumann algebra. An O*-algebra $\mathcal{M}$ is said to be *semifinite* if $(\mathcal{M}'_{\mathrm{w}})'$ is a semifinite von Neumann algebra. For the general theory of O*-algebras we refer to [10, 14]. We introduce the notion of generalized vectors for O*-algebras. Let $\mathcal{M}$ be a closed O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$. A *generalized vector* λ for $\mathcal{M}$ is a linear map of the left ideal $\mathcal{D}(\lambda)$ of $\mathcal{M}$ into $\mathcal{D}$ satisfying $\lambda(XY) = X\lambda(Y)$ for each $X \in \mathcal{M}$ and $Y \in \mathcal{D}(\lambda)$; and a generalized vector λ is said to be *tracial* if $(\lambda(X)|\lambda(Y)) = (\lambda(Y^{\dagger})|\lambda(X^{\dagger}))$ for each $X, Y \in \mathcal{D}(\lambda)^{\dagger} \cap \mathcal{D}(\lambda)$. Let λ be a generalized vector for $\mathcal{M}$. Suppose

(S)$_1$ $\lambda((\mathcal{D}(\lambda)^{\dagger} \cap \mathcal{D}(\lambda))^2)$ is total in $\mathcal{H}$. Then we define two commutants λ' and λ^C of λ as follows:

$$\begin{cases} \mathcal{D}(\lambda') = \{C \in \mathcal{M}'_{\mathrm{w}}; {}^{\exists}\xi_C \in \displaystyle\bigcap_{X \in \mathcal{D}(X)} \mathcal{D}(\overline{X}) \text{ s.t. } C\lambda(X) = \overline{X}\xi_C, {}^{\forall}X \in \mathcal{D}(\lambda)\}, \\ \lambda'(C) = \xi_C, \quad C \in \mathcal{D}(\lambda'), \end{cases}$$

$$\begin{cases} \mathcal{D}(\lambda^C) = \{C \in \mathcal{M}'_{\mathrm{w}} \ ; \ {}^{\exists}\xi_C \in \mathcal{D} \text{ s.t. } C\lambda(X) = X\xi_C, \ {}^{\forall}X \in \mathcal{D}(\lambda)\}, \\ \lambda^C(C) = \xi_C, \quad C \in \mathcal{D}(\lambda^C), \end{cases}$$

and then λ' and λ^C are generalized vectors for the von Neumann algebra $\mathcal{M}'_{\mathrm{w}}$ such that $\lambda^C \subset \lambda'$, that is, $\mathcal{D}(\lambda^C) \subset \mathcal{D}(\lambda')$ and $\lambda^C(C) = \lambda'(C)$, ${}^{\forall}C \in \mathcal{D}(\lambda^C)$.

Definition 2.1. Let λ be a generalized vector for $\mathcal{M}$. Suppose λ satisfies the above condition (S)$_1$ and the following condition (S)$'_2$ (resp. (S)$^C_2$). Then $(\mathcal{M}, \lambda, \lambda')$ (resp. $(\mathcal{M}, \lambda, \lambda^C)$) is said to be a cyclic and separating system:
(S)$'_2$ $\lambda'((\mathcal{D}(\lambda')^* \cap \mathcal{D}(\lambda'))^2)$ is total in $\mathcal{H}$.
(S)$^C_2$ $\lambda^C((\mathcal{D}(\lambda^C)^* \cap \mathcal{D}(\lambda^C))^2)$ is total in $\mathcal{H}$.

When $(\mathcal{M}, \lambda, \lambda^C)$ is a cyclic and separating system, λ is said to be a cyclic and separating generalized vector.

Suppose $(\mathcal{M}, \lambda, \lambda')$ is a cyclic and separating system and put

$$\begin{cases} \mathcal{D}(\lambda'') = \{A \in (\mathcal{M}'_{\mathrm{w}})' \ ; \ {}^{\exists}\xi_A \in \mathcal{H} \text{ s.t. } A\lambda'(C) = C\xi_A, \ {}^{\forall}C \in \mathcal{D}(\lambda')\}, \\ \lambda''(A) = \xi_A, \ A \in \mathcal{D}(\lambda''). \end{cases}$$

Then λ'' is a cyclic and separating generalized vector for the von Neumann algebra $(\mathcal{M}'_{\mathrm{w}})'$, so that the maps $\lambda(X) \to \lambda(X^{\dagger}), X \in \mathcal{D}(\lambda)^{\dagger} \cap \mathcal{D}(\lambda)$ and $\lambda''(A) \to \lambda''(A^*), A \in \mathcal{A}(\lambda'')^* \cap \mathcal{D}(\lambda'')$ are closable in $\mathcal{H}$ and their closures are denoted by S_λ and $S_{\lambda''}$, respectively. Let $S_\lambda = J_\lambda \Delta_\lambda^{\frac{1}{2}}$ and $S_{\lambda''} = J_{\lambda''}\Delta_{\lambda''}^{\frac{1}{2}}$ be the polar decompositions of S_λ and $S_{\lambda''}$, respectively. Then it is shown that $S_\lambda \subset S_{\lambda''}$, and $J_{\lambda''}(\mathcal{M}'_{\mathrm{w}})'J_{\lambda''} = \mathcal{M}'_{\mathrm{w}}$ and $\Delta_{\lambda''}^{it}(\mathcal{M}'_{\mathrm{w}})'\Delta_{\lambda''}^{-it} = (\mathcal{M}'_{\mathrm{w}})'$ for all $t \in \mathbb{R}$ by the Tomita fundamental theorem. But, we don't know how the unitary group $\{\Delta_{\lambda''}^{it}\}_{t\in\mathbb{R}}$ acts on the O*-algebra $\mathcal{M}$, and so we define a system which has the best condition:

Definition 2.2. A cyclic and separating system $(\mathcal{M}, \lambda, \lambda')$ is said to be standard if the following conditions (S)$'_3$ and (S)$'_4$ hold:

(S)$'_3$ $\Delta^{it}_{\lambda''}\mathcal{D} \subset \mathcal{D}$ and $\Delta^{it}_{\lambda''}\mathcal{M}\Delta^{-it}_{\lambda''} = \mathcal{M}, \ ^\forall t \in \mathbb{R}$.
(S)$'_4$ $\Delta^{it}_{\lambda''}(\mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda))\Delta^{-it}_{\lambda''} = \mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda), \ ^\forall t \in \mathbb{R}$.

By ([7], Theorem 5.5) we have the following

Theorem 2.3. *Suppose $(\mathcal{M}, \lambda, \lambda')$ is a standard system. Then the following statements hold:*

(1) *$S_\lambda = S_{\lambda''}$, and so $J_\lambda = J_{\lambda''}$ and $\Delta_\lambda = \Delta_{\lambda''}$.*

(2) *$\{\sigma^\lambda_t\}_{t\in\mathbb{R}}$ is a one-parameter group of $*$-automorphisms of $\mathcal{M}$, where $\sigma^\lambda_t(X) \equiv \Delta^{it}_\lambda X \Delta^{-it}_\lambda, \ X \in \mathcal{M}, t \in \mathbb{R}$.*

(3) *λ satisfies the KMS-condition with respect to $\{\sigma^\lambda_t\}_{t\in\mathbb{R}}$; that is, for any $X, Y \in \mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda)$ there exists an element $f_{X,Y}$ of $A(0,1)$ such that $f_{X,Y}(t) = (\lambda(\sigma^\lambda_t(X))|\lambda(Y))$ and $f_{X,Y}(t+i) = (\lambda(Y^\dagger)|\lambda(\sigma^\lambda_t(X^\dagger)))$ for all $t \in \mathbb{R}$, where $A(0,1)$ is the set of all complex-valued functions, bounded and continuous on $0 \leq \mathrm{Im} z \leq 1$ and analytic in the interior.*

Suppose $(\mathcal{M}, \lambda, \lambda^C)$ is a cyclic and separating system and put

$$\begin{cases} \mathcal{D}(\lambda^{CC}) = \{A \in (\mathcal{M}'_{\mathrm{w}})' \ ; \ ^\exists \xi_A \in \mathcal{H} \text{ s.t } A\lambda^C(C) = C\xi_A, \ ^\forall C \in \mathcal{D}(\lambda^C)\}, \\ \lambda^{CC}(A) = \xi_A, \ A \in \mathcal{D}(\lambda^{CC}). \end{cases}$$

Then λ^{CC} is a cyclic and separating generalized vector for the von Neumann algebra $(\mathcal{M}'_{\mathrm{w}})'$, and so the closure of the closable involution $\lambda^{CC}(A) \to \lambda^{CC}(A^*), A \in \mathcal{D}(\lambda^{CC})^* \cap \mathcal{D}(\lambda^{CC})$, is denoted by $S_{\lambda^{CC}}$ and let $S_{\lambda^{CC}} = J_{\lambda^{CC}}\Delta^{\frac{1}{2}}_{\lambda^{CC}}$ be the polar decomposition of $S_{\lambda^{CC}}$. Since $\lambda^C \subset \lambda'$, it follows that $(\mathcal{M}, \lambda, \lambda')$ is a cyclic and separating system and $S_\lambda \subset S_{\lambda''} \subset S_{\lambda^{CC}}$.

Definition 2.4. A cyclic and separating system $(\mathcal{M}, \lambda, \lambda^C)$ is said to be standard if the following conditions (S)$^C_3$ and (S)$^C_4$ hold:
(S)$^C_3$ $\Delta^{it}_{\lambda^{CC}}\mathcal{D} \subset \mathcal{D}$ and $\Delta^{it}_{\lambda^{CC}}\mathcal{M}\Delta^{-it}_{\lambda^{CC}} = M, ^\forall t \in \mathbb{R}$.
(S)$^C_4$ $\Delta^{it}_{\lambda^{CC}}(\mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda))\Delta^{-it}_{\lambda^{CC}} = \mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda), \ ^\forall t \in \mathbb{R}$.

By ([7], Theorem 5.6) we have the following

Theorem 2.5. *Suppose λ is a standard generalized vector for $\mathcal{M}$. Then $(\mathcal{M}, \lambda, \lambda')$ is a standard system satisfying $S_\lambda = S_{\lambda''} = S_{\lambda^{CC}}$.*

Any element ξ of $\mathcal{D}$ is regarded as generalized vector for $\mathcal{M}$ by $\mathcal{D}(\lambda_\xi) = \mathcal{M}$ and $\lambda_\xi(X) = X\xi, X \in \mathcal{M}$. It is easily shown that if ξ is *cyclic*, that is, $\mathcal{M}\xi$ is dense in $\mathcal{H}$ (iff λ_ξ is cyclic), then $\mathcal{D}(\lambda^C_\xi) = \mathcal{D}(\lambda'_\xi) = \mathcal{M}'_{\mathrm{w}}$ and $\lambda^C_\xi(C) = \lambda'_\xi(C) = C\xi, ^\forall C \in \mathcal{M}'_{\mathrm{w}}$. When both $\mathcal{M}\xi$ and $\mathcal{M}'_{\mathrm{w}}\xi$ are dense in $\mathcal{H}$, we simply denote S_{λ_ξ}, J_{λ_ξ} and Δ_{λ_ξ} (resp. $S_{\lambda''_\xi}$, $J_{\lambda''_\xi}$ and $\Delta_{\lambda''_\xi}$) by S_ξ, J_ξ and Δ_ξ (resp. S''_ξ, J''_ξ and Δ''_ξ), respectively. If λ_ξ is standard, then ξ is said to be a *standard vector* for $\mathcal{M}$. If λ_ξ is tracial, then ξ is said to be a *tracial vector* for $\mathcal{M}$.

For the standardness of tracial generalized vectors we have the following

Proposition 2.6. *Let $\mathcal{M}$ be a closed O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$. Suppose μ is a tracial generalized vector for $\mathcal{M}$ such that $\mu((\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))^2)$ is total in $\mathcal{H}$. Then the following statements are equivalent:*

(i) *μ is standard; that is, $(\mathcal{M}, \mu, \mu^C)$ is a standard system.*
(ii) *$\mu^C((\mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C))^2)$ is total in $\mathcal{H}$.*

(iii) $(\mathcal{M}, \mu, \mu')$ *is a standard system.*

(iv) $\mu'((\mathcal{D}(\mu')^* \cap \mathcal{D}(\mu'))^2)$ *is total in* $\mathcal{H}$.

(v) $J_\mu(\mathcal{M}'_{\mathrm{w}})' J_\mu = \mathcal{M}'_{\mathrm{w}}$, *where* J_μ *is the unitary involution on* $\mathcal{H}$ *defined by* $J_\mu \mu(Y) = \mu(Y^\dagger)$ *for each* $Y \in \mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu)$.

If this is true, then $J_\mu = J_{\mu^{CC}} = J_{\mu''}$ *and* $\Delta_\mu = \Delta_{\mu^{CC}} = \Delta_{\mu''} = I$, *and further* $Y^{\dagger *} = \overline{Y}$ *for each* $Y \in \mathcal{D}(\mu)$.

Proof. We show the equivalence of (i) $\sim$ (v) by (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii). We show the implication (v) $\Rightarrow$ (ii). The other implications are trivial. Let $X \in \mathcal{M}$, and let $\overline{X} = U_X|\overline{X}|$, the polar decomposition of $\overline{X}$, and $|\overline{X}| = \int_0^\infty t\, dE_X(t)$, the spectral resolution of $|\overline{X}|$. We put$E_X(n) = \int_0^n dE_X(t)$ and $X_n = \overline{X}E_X(n)$, $n \in \mathbb{N}$. Then we have $U_X, E_X(n), X_n \in (\mathcal{M}'_{\mathrm{w}})'$ for each $n \in \mathbb{N}$. Take arbitrary $Y \in \mathcal{D}(\mu)$ and $n \in \mathbb{N}$. Then, it is not difficult to show

(2.1)
$$\begin{cases} J_\mu Y_n J_\mu \in \mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C), \mu^C(J_\mu Y_n J_\mu) = J_\mu U_Y E_Y(n) U_Y^* J_\mu \mu(Y^\dagger), \\ \mu^C(J_\mu Y_n^* J_n) = J_\mu E_Y(n) J_\mu \mu(Y), \end{cases}$$

$$\lim_{n\to\infty} \mu^C(J_\mu Y_n J_\mu) = \mu(Y^\dagger). \tag{2.2}$$

Hence it follows that $\mu^C((\mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C))^2)$ is total in $\mathcal{H}$ since $\mu((\mathcal{D}(\mu)^* \cap \mathcal{D}(\mu))^2)$ is total in $\mathcal{H}$. Thus the statements (i) $\sim$ (v) are equivalent and $\Delta_{\mu''} = \Delta_{\mu^{CC}} = I$ and $J_{\mu''} = J_{\mu^{CC}} = J_\mu$.

We finally show that $Y^{\dagger *} = \overline{Y}$ for each $Y \in \mathcal{D}(\mu)$. By (2.1) and (2.2) we can show by simple calculations

$$\begin{gathered} \mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C) = \{J_\mu A^* J_\mu \ ; \ A \in \mathfrak{A}\}, \\ \mu^C(J_\mu A^* J_\mu) = \mu^{CC}(A), \quad \mu^C(J_\mu A J_\mu) = \mu^{CC}(A^*), \ A \in \mathfrak{A}, \end{gathered} \tag{2.3}$$

where $\mathfrak{A} \equiv \{A \in \mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC}) \ ; \ \mu^{CC}(A), \mu^{CC}(A^*) \in \mathcal{D}\}$. Take an arbitrary $Y \in \mathcal{D}(\mu)$. By (2.3) and (2.2) we have

$$\pi_0(\mu(Y))\mu^{CC}(A) \equiv J_\mu A^* J_\mu \mu(Y) = Y\mu^{CC}(A),$$

$$\begin{aligned} \pi_0(J_\mu\mu(Y))\mu^{CC}(A) &\equiv J_\mu A^* \mu(Y) = \lim_{n\to\infty} J_\mu A^* \mu^C(J_\mu Y_n^* J_\mu) \\ &= \lim_{n\to\infty} Y_n^* \mu^{CC}(A) = Y^\dagger \mu^{CC}(A) \end{aligned}$$

for each $A \in \mathfrak{A}$, and further since $\mathfrak{A}$ is a Hilbert algebra in $\mathcal{H}$ by (2.3), it follows that $\overline{\pi_0(\mu(Y))} \subset \overline{Y}$ and $\overline{\pi_0(J_\mu\mu(Y))} \subset \overline{Y^\dagger}$. Further, it follows from the theory of Hilbert algebra [9] that $\pi_0(\xi)^* = \overline{\pi_0(J_\mu\xi)}$, $^\forall\xi \in \mathcal{H}$, that $\overline{\pi_0(\mu(Y))} \subset \overline{Y} \subset Y^{\dagger *} \subset \pi_0(J_\mu\mu(Y))^* = \overline{\pi_0(\mu(Y))}$. Hence we have $Y^{\dagger *} = \overline{Y} = \overline{\pi_0(\mu(Y))}$ for each $Y \in \mathcal{D}(\mu)$. This completes the proof. □

By Proposition 2.6 we have the following

Corollary 2.7. *Let* $\mathcal{M}$ *be a closed* O^**-algebra on* $\mathcal{D}$ *in* $\mathcal{H}$ *such that* $\mathcal{M}'_{\mathrm{w}}\mathcal{D} \subset \mathcal{D}$ *and* $\xi_0 \in \mathcal{D}$. *Suppose* ξ_0 *is a cyclic tracial vector for* $\mathcal{M}$. *Then the following statements* *are equivalent:*

(i) ξ_0 *is standard.*
(ii) $\mathcal{M}'_{\rm w}\xi_0$ *is dense in* $\mathcal{H}$.
(iii) $J_{\xi_0}(\mathcal{M}'_{\rm w})'J_{\xi_0} = \mathcal{M}'_{\rm w}$.

If this is true, then $J_{\xi_0} = J''_{\xi_0}$, $\Delta_{\xi_0} = \Delta''_{\xi_0} = I$ *and* $\mathcal{M}$ *is an integrable O*-algebra on* $\mathcal{D}$. *Further,* $\overline{\mathcal{M}} \equiv \{\overline{X}\ ;\ X \in \mathcal{M}\}$ *is a* $*$*-subalgebra of the* $*$*-algebra* $L^\omega(\omega_{\xi_0}) \equiv \bigcap_{1\leq p<\infty} L^p(\omega_{\xi_0})$ *equipped with the strong sum, strong scalar multiplication, strong product and adjoint, where* $L^p(\omega_{\xi_0})$ *is the Segal* L^p*-space with respect to the vector trace* ω_{ξ_0} *on* $(\mathcal{M}'_{\rm w})'$ *(refer to* [4]*).*

3. Standard systems for semifinite O*-algebras

In this section we treat a standard system $(\mathcal{M}, K'\mu, (K'\mu)')$ constructed by a standard tracial generalized vector μ and a non-singular positive self-adjoint operator K', and consider when a standard system $(\mathcal{M}, \lambda, \lambda')$ is unitarily equivalent to such a standard system $(\mathcal{N}, K'\mu, (K'\mu)')$. Let $\mathcal{M}$ be a closed O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_{\rm w}\mathcal{D} \subset \mathcal{D}$, μ a standard tracial generalized vector for $\mathcal{M}$ and K' a non-singular positive self-adjoint operator in $\mathcal{H}$ affiliated with $\mathcal{M}'_{\rm w}$ whose domain $\mathcal{D}(K')$ contains $\mu(\mathcal{D}(\mu))$. Let $K' = \int_0^\infty t dE'(t)$ and $K \equiv J_\mu K' J_\mu = \int_0^\infty t dE(t)$ be the spectral resolutions of K' and K, respectively and let $E'(n) = \int_0^n dE'(t)$ and $E(n) = \int_0^n dE(t)$ for $n \in \mathbb{N}$. Here we put

$$\begin{cases} \mathcal{D}(K'\mu) = \mathcal{D}(\mu), \\ (K'\mu)(X) = K'\mu(X), \quad X \in \mathcal{D}(\mu). \end{cases}$$

Then it is easily shown that $K'\mu$ is a generalized vector for $\mathcal{M}$. For the standardness of the system $(\mathcal{M}, K'\mu, (K'\mu)')$ we have the following

Proposition 3.1. *Let* $\mathcal{M}$ *be a closed O*-algebra on* $\mathcal{D}$ *in* $\mathcal{H}$ *such that* $\mathcal{M}'_{\rm w}\mathcal{D} \subset \mathcal{D}$ *and* μ *a standard tracial generalized vector for* $\mathcal{M}$. *Suppose* K' *is a non-singular positive self-adjoint operator in* $\mathcal{H}$ *affiliated with* $\mathcal{M}'_{\rm w}$ *such that* $\mu(\mathcal{D}(\mu)) \subset \mathcal{D}(K')$, $K'\mu((\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))^2)$ *is total in* $\mathcal{H}$ *and* $K'\mu(\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))$ *is dense in the Hilbert space* $\mathcal{D}(K \cdot K'^{-1})$. *Then* $K'\mu$ *is a generalized vector for* $\mathcal{M}$ *satisfying the following conditions:*

(i) $(K'\mu)'((\mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'))^2)$ *is total in* $\mathcal{H}$.
(ii) $S_{K'\mu} = S_{(K'\mu)''} = J_\mu K \cdot K'^{-1}$.

Further, $(\mathcal{M}, K'\mu, (K'\mu)')$ *is a standard system if and only if* $K^{it}\mathcal{D} \subset \mathcal{D}$ *and* $K^{it}YK^{-it}\lceil\mathcal{D} \in \mathcal{D}(\mu)$ *for all* $Y \in \mathcal{D}(\mu)$ *and* $t \in \mathbb{R}$.

Proof. Since μ is standard, there exists a net $\{A_\alpha\}$ in $\mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC})$ which converges strongly* to I. Let $Y \in \mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu)$ and let $Y_n = \overline{Y}E_Y(n), n \in \mathbb{N}$. Then it follows from ([7], Lemma 5.2) that $\{Y_n\} \subset \mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC}), Y_n \to Y$ strongly*, $\lim_{n\to\infty} \mu(Y_n) = \mu(Y)$ and $\lim_{n\to\infty} \mu^{CC}(Y_n^*) = \mu(Y^\dagger)$. Take arbitrary $C \in \mathcal{D}(\mu^C)$, $Y \in$

$\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu)$ and $m, n \in \mathbb{N}$. Then we have

$$\begin{aligned} E'(n)CE'(m)(K'\mu)(Y) &= \lim_{k\to\infty} E'(n)CE'(m)K'\mu^{CC}(Y_k) \\ &= \lim_{k\to\infty} E'(n)CJ_\mu KE(m)\mu^{CC}(Y_k^*) \\ &= \lim_{k\to\infty}\lim_{\alpha} E'(n)CJ_\mu A_\alpha KE(m)\mu^{CC}(Y_k^*) \\ &= \lim_{k\to\infty}\lim_{\alpha} E'(n)CJ_\mu \mu^{CC}(A_\alpha KE(m)Y_k^*) \\ &= \lim_{k\to\infty}\lim_{\alpha} E'(n)C\mu^{CC}(Y_k KE(m)A_\alpha^*) \\ &= \lim_{k\to\infty}\lim_{\alpha} Y_k E'(n)KE(m)A_\alpha^*\mu^{C}(C) \\ &= \lim_{k\to\infty} Y_k E'(n)KE(m)\mu^{C}(C), \end{aligned}$$

and so

$$\begin{aligned} (Y^\dagger\eta | E'(n)KE(m)\mu^C(C)) &= \lim_{k\to\infty}(Y_k^*\eta | E'(n)KE(m)\mu^C(C)) \\ &= \lim_{k\to\infty}(\eta | Y_k E'(n)KE(m)\mu^C(C)) \\ &= (\eta | E'(n)CE'(m)(K'\mu)(Y)), \end{aligned}$$

which implies $E'(n)KE(m)\mu^C(C) \in \mathcal{D}(Y^{\dagger *}) = \mathcal{D}(\overline{Y})$,

$$\overline{Y}E'(n)KE(m)\mu^C(C) = E'(n)CE'(m)(K'\mu)(Y).$$

Hence we have

$$\begin{aligned} &E'(n)CE'(m) \in \mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'), \\ &(K'\mu)'(E'(n)CE'(m)) = E'(n)KE(m)\mu^C(C), \\ &\qquad\qquad {}^\forall C \in \mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C), {}^\forall m, n \in \mathbb{N}. \end{aligned} \tag{3.1}$$

Similarly, we have

$$\begin{aligned} &E'(n)CE'(m)K'^{-1} \in \mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'), \\ &(K'\mu)'(E'(n)CE'(m)K'^{-1}) = E'(n)E(m)\mu^C(C), \\ &(K'\mu)'(E'(m)K'^{-1}C^*E'(n)) = E'(m)K'^{-1}E(n)K\mu^C(C) \\ &\qquad\qquad {}^\forall C \in \mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C),\ m, n \in \mathbb{N}. \end{aligned} \tag{3.2}$$

By (3.1) and (3.2) we have

$$E'(n)C_1E'(l)C_2E'(m)K'^{-1} \in (\mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'))^2,$$

$$\begin{aligned} &\lim_{m,n\to\infty}(K'\mu)'(E'(n)C_1E'(l)C_2E'(m)K'^{-1}) \\ &\qquad = \lim_{m,n\to\infty}(E'(n)C_1E'(l)(K'\mu)'(E'(l)C_2E'(m)K'^{-1}) \\ &\qquad = \lim_{m,n\to\infty} E'(n)C_1E'(l)E(m)\mu^C(C_2) \\ &\qquad = \mu^C(C_1C_2) \end{aligned}$$

for each $C_1, C_2 \in \mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C)$, which implies since $\mu^C((\mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C))^2)$ is total in $\mathcal{H}$ that $(K'\mu)'(((\mathcal{D}(K'\mu)')^* \cap \mathcal{D}((K'\mu)'))^2)$ is total in $\mathcal{H}$. We show $J_\mu K \cdot$

$K'^{-1} = S_{K'\mu} = S_{(K'\mu)''}$. Since

(3.3)
$$K'\mu(\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu)) \text{ is densely contained in the Hilbert space } \mathcal{D}(K \cdot K'^{-1}),$$
$$J_\mu K \cdot K'^{-1} K'\mu(Y) = J_\mu K\mu(Y) = K'\mu(Y^\dagger) = S_{K'\mu}(K'\mu)(Y)$$

for each $Y \in \mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu)$, we have $J_\mu K \cdot K'^{-1} \subset S_{K'\mu}$. We generally have $S_{K'\mu} \subset S_{(K'\mu)''}$, and hence $J_\mu K \cdot K'^{-1} \subset S_{K'\mu} \subset S_{(K'\mu)''}$. Conversely we show $S_{(K'\mu)''} \subset J_\mu K \cdot K'^{-1}$. It is shown similarly to (3.1) that

$$E'(n)CE'(m) \in \mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'),$$
$$(K'\mu)'(E'(n)CE'(m)) = E'(n)KE'(m)\mu^{CCC}(C), \quad {}^\forall C \in \mathcal{D}(\mu^{CCC})^* \cap \mathcal{D}(\mu^{CCC}), \quad {}^\forall m, n \in \mathbb{N},$$
$$J_\mu A^* J_\mu \in \mathcal{D}(\mu^{CCC}) \text{ and } \mu^{CCC}(J_\mu A^* J_\mu) = \mu^{CC}(A), \quad {}^\forall A \in \mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC}),$$

so that by (3.1)

$$E'(n)J_\mu A^* J_\mu E'(m) \in \mathcal{D}((K'\mu)')^* \cap \mathcal{D}((K'\mu)'),$$
$$(K'\mu)'(E'(n)J_\mu A^* J_\mu E'(m)) = KE(m)E'(n)\mu^{CC}(A), \quad {}^\forall A \in \mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC}).$$

Hence we have

$$\begin{aligned}
(KK'^{-1}K'E'(n)E(m)\mu^{CC}(A)|(K'\mu)''(B))
&= ((K'\mu)'(E'(n)J_\mu A^* J_\mu E'(m))|(K'\mu)''(B)) \\
&= (S_{(K'\mu)''}(K'\mu)''(B)|S^*_{(K'\mu)''}(K'\mu)'(E'(n)J_\mu A^* J_\mu E'(m))) \\
&= (S_{(K'\mu)''}(K'\mu)''(B)|(K'\mu)'(E'(m)J_\mu A J_\mu E'(n))) \\
&= (S_{(K'\mu)''}(K'\mu)''(B)|KE(n)E'(m)\mu^{CC}(A^*)) \\
&= (S_{(K'\mu)''}(K'\mu)''(B)|J_\mu K'E'(n)E(m)\mu^{CC}(A)) \\
&= (K'E'(n)E(m)\mu^{CC}(A)|J_\mu S_{(K'\mu)''}(K'\mu)''(B))
\end{aligned}$$

for each $A \in \mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC})$ and $B \in \mathcal{D}((K'\mu)'')^* \cap \mathcal{D}((K'\mu)'')$, and further it follows from (3.3) that $\{K'E'(n)E(m)\mu^{CC}(A)\ ;\ A \in \mathcal{D}(\mu^{CC})^* \cap \mathcal{D}(\mu^{CC}) \text{ and } m, n \in \mathbb{N}\}$ is total in the Hilbert space $\mathcal{D}(K \cdot K'^{-1})$, which implies $S_{(K'\mu)''} \subset J_\mu K \cdot K'^{-1}$. Thus we have $S_{K'\mu} = S_{(K'\mu)''} = J_\mu K \cdot K'^{-1}$, and hence

(3.4) $$J_{K'\mu} = J_{(K'\mu)''} = J_\mu \text{ and } \Delta_{K'\mu} = \Delta_{(K'\mu)''} = K \cdot K'^{-1}.$$

It follows from (3.4) that $(\mathcal{M}, K'\mu, (K'\mu)')$ is a standard system if and only if $K^{it}\mathcal{D} \subset \mathcal{D}$ and $K^{it} Y K^{-it} \lceil \mathcal{D} \in \mathcal{D}(\mu)$ for all $Y \in \mathcal{D}(\mu)$ and $t \in \mathbb{R}$. This completes the proof. □

We consider when the condition in Proposition 3.1 $K^{it}\mathcal{D} \subset \mathcal{D}$ and $K^{it} Y K^{-it} \lceil \mathcal{D} \in \mathcal{D}(\mu)$ for all $Y \in \mathcal{D}(\mu)$ and $t \in \mathbb{R}$ holds.

Corollary 3.2. *Let $(\mathcal{M}, \mu, K')$ be given in Proposition 3.1. Suppose μ is full, that is, $\mathcal{D}(\mu) = \{X \in \mathcal{M};\ {}^\exists \xi_X \in \mathcal{D} \text{ s.t.} X\lambda^C(C) = C\xi_X,\ {}^\forall C \in \mathcal{D}(\lambda^C)^* \cap \mathcal{D}(\lambda^C)\}$, and $K^{it} \lceil \mathcal{D} \in \mathcal{M}$ for all $t \in \mathbb{R}$. Then $(\mathcal{M}, K'\mu, (K'\mu)')$ is a standard system.*

Proof. Take arbitrary $Y \in \mathcal{D}(\mu)$ and $t \in \mathbb{R}$. Then we have

$$\begin{aligned}(K^{it}YK^{-it}\mu^C(C)|\xi) &= (J_\mu K'^{-it}\mu^C(C^*)|Y^\dagger K^{-it}\xi)\\ &= \lim_\alpha (J_\mu C_\alpha K'^{-it}\mu^C(C^*)|Y^\dagger K^{-it}\xi)\\ &= \lim_\alpha (Y\mu^C(CK'^{-it}C_\alpha^*)|K^{-it}\xi)\\ &= \lim_\alpha (CK'^{-it}C_\alpha^*\mu(Y)|K^{-it}\xi)\\ &= (CK^{it}K'^{-it}\mu(Y)|\xi)\end{aligned}$$

for all $C \in \mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C)$ and $\xi \in \mathcal{D}$, where $\{C_\alpha\}$ is a net in $\mathcal{D}(\mu^C)^* \cap \mathcal{D}(\mu^C)$ which converges strongly* to I. Hence it follows from the fullness of μ that $K^{it}YK^{-it} \in \mathcal{D}(\mu)$. By Proposition 3.1 $(\mathcal{M}, K'\mu, (K'\mu)')$ is a standard system. □

We next consider the converse of Proposition 3.1:

When is a standard system $(\mathcal{M}, \lambda, \lambda')$ unitarily equivalent to such a standard system $(\mathcal{N}, K'\mu, (K'\mu)')$ in Proposition 3.1?

Proposition 3.3. *Let $\mathcal{M}$ be a closed semifinite O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_{\rm w}\mathcal{D} \subset \mathcal{D}$. Suppose λ is a generalized vector for $\mathcal{M}$ such that*

(i) $\lambda((\mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda))^2)$ *is total in $\mathcal{H}$;*
(ii) $\lambda'((\mathcal{D}(\lambda')^* \cap \mathcal{D}(\lambda'))^2)$ *is total in $\mathcal{H}$;*
(iii) $S_\lambda = S_{\lambda''}$;
(iv) $\overline{Y} \in L^2(\tau'')$ *for each $Y \in \mathcal{D}(\lambda)$, where τ'' is a faithful normal semifinite trace on $(\mathcal{M}'_{\rm w})'$.*

Then there exist a standard tracial generalized vector μ for a closed O^*-algebra $\mathcal{N}$ in $L^2(\tau'')$ and a non-singular positive self-adjoint operator K' in $L^2(\tau'')$ affiliated with $\mathcal{N}'_{\rm w}$ such that (μ, K') satisfies all of the conditions in Proposition 3.1 and λ is unitarily equivalent to the generalized vector $K'\mu$; that is, there exists a unitary operator U of $L^2(\tau'')$ onto $\mathcal{H}$ such that $U^*\mathcal{M}U = \mathcal{N}, U^*\mathcal{D}(\lambda)U = \mathcal{D}(\mu)$ and $\lambda(Y) = U(K'\mu)(U^*YU)$ for each $Y \in \mathcal{D}(\lambda)$.

Proof. By the assumption for λ, $\lambda''(\mathcal{D}(\lambda'')^* \cap \mathcal{D}(\lambda''))$ is an achieved left Hilbert algebra in $\mathcal{H}$ whose left von Neumann algebra equals the semifinite von Neumann algebra $(\mathcal{M}'_{\rm w})'$, so that the following results have been shown by Takesaki [15] :

(3.5) We put $\Pi_0\lambda''(B) = B$, $B \in \mathcal{D}(\lambda'') \cap \mathfrak{N}_{\tau''}$. Then Π_0 is a closable operator of the dense subspace $\lambda''(\mathcal{D}(\lambda'') \cap \mathfrak{N}_{\tau''})$ onto the dense subspace $\mathcal{D}(\lambda'') \cap \mathfrak{N}_{\tau''}$ in $L^2(\tau'')$ whose closure Π is non-singular.

(3.6) Let $\Pi = VT'$ be the polar decomposition of Π. Then V is a unitary operator of $\mathcal{H}$ onto $L^2(\tau'')$ and T' is a non-singular positive self-adjoint operator in $\mathcal{H}$ affiliated with $\mathcal{M}'_{\rm w}$ such that $\Delta_{\lambda''}^{\frac{1}{2}} = T^{-1} \cdot T'$, where $T = J_{\lambda''}T'J_{\lambda''}$.

(3.7) Let ρ_0 be the left regular representation of $(\mathcal{M}'_{\rm w})'$ on $L^2(\tau'')$ defined by $\rho_0(A)B = AB$, $A \in (\mathcal{M}'_{\rm w})'$, $B \in \mathfrak{N}_{\tau''}$. Then the unitary operator V implements a spatial isomorphism between $(\mathcal{M}'_{\rm w})'$ and $\rho_0((\mathcal{M}'_{\rm w})')$ such that $VBV^* = \rho_0(B)$ for each $B \in \mathcal{D}(\lambda'') \cap \mathfrak{N}_{\tau''}$.

(3.8) $\lambda''(\mathcal{D}(\lambda'')^* \cap \mathcal{D}(\lambda'') \cap \mathfrak{N}_{\tau''})$ is dense in the Hilbert space $\mathcal{D}(T')$.

Let Λ be the inverse of Π and $\Lambda = UK'$ be the polar decomposition of Λ. Then we have $U = V^*$ and $K' = U^*T'^{-1}U$. It follows from (3.6) that U is a unitary operator of $L^2(\tau'')$ onto $\mathcal{H}$ and K' is a non-singular positive self-adjoint operator in $L^2(\tau'')$ affiliated with the von Neumann algebra $\rho_0((\mathcal{M}'_{\rm w})')'$. We put

$$\mathcal{N} = U^*\mathcal{M}U,$$
$$\mathcal{D}(\mu) = U^*\mathcal{D}(\lambda)U \text{ and } \mu(U^*YU) = \overline{Y} \in L^2(\tau''), \quad Y \in \mathcal{D}(\lambda).$$

Then $\mathcal{N}$ is a closed O*-algebra on $U^*\mathcal{D}$ in $L^2(\tau'')$ such that $\mathcal{N}'_{\rm w} = U^*\mathcal{M}'_{\rm w}U$ and $(\mathcal{N}'_{\rm w})' = U^*(\mathcal{M}'_{\rm w})'U = \rho_0((\mathcal{M}'_{\rm w})')$, and by (3.7) μ is a tracial generalized vector for $\mathcal{N}$. We show

$$\lambda(Y) \in \mathcal{D}(\Pi) \text{ and } \Pi\lambda(Y) = \mu(U^*YU),\ Y \in \mathcal{D}(\lambda).$$

In fact, let $X \in \mathcal{M}$ and let $\overline{X} = U_X\left|\overline{X}\right|$ be the polar decomposition of $\overline{X}$ and $\left|\overline{X}\right| = \int_0^\infty t dE_X(t)$ the spectral resolution of $\left|\overline{X}\right|$. Take an arbitrary $Y \in \mathcal{D}(\lambda)$. Then it is shown that $\overline{Y}E_Y(n) \in \mathcal{D}(\lambda'') \cap \mathfrak{N}_{\tau''}$ and $\lambda''(\overline{Y}E_Y(n)) = E_{Y^\dagger}(n)\lambda(Y)$, $n \in \mathbb{N}$. Hence we have

$$\lim_{n\to\infty}\|E_{Y^\dagger}(n)\lambda(Y) - \lambda(Y)\| = 0,$$
$$\lim_{n\to\infty}\|\Pi E_{Y^\dagger}(n)\lambda(Y) - \mu(U^*YU)\| = \lim_{n\to\infty}\tau''((I - E_Y(n))|\overline{Y}|^2) = 0,$$

which implies $\lambda(Y) \in \mathcal{D}(\Pi)$ and $\Pi\lambda(Y) = \mu(U^*YU)$. Hence we have $\mu(\mathcal{D}(\mu)) \subset \mathcal{D}(K')$. Since $K'\mu(U^*YU) = U^*\lambda(Y)$ for each $Y \in \mathcal{D}(\lambda)$ and $\lambda((\mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda))^2)$ is total in $\mathcal{H}$, it follows that $(K'\mu)((\mathcal{D}(K'\mu)^\dagger \cap \mathcal{D}(K'\mu))^2)$ is total in $L^2(\tau'')$. Further, since we have

$$(K'\mu)(U^*YU) = U^*\lambda(Y),$$
$$(K \cdot K'^{-1})(K'\mu)(U^*YU) = U^*T^{-1} \cdot T'\lambda(Y) = U^*J_\lambda\lambda(Y^\dagger)$$

for each $Y \in \mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda)$, and further $S_\lambda = S_{\lambda''} = J_\lambda T^{-1} \cdot T'$ by (3.6) and $\lambda(\mathcal{D}(\lambda)^\dagger \cap \mathcal{D}(\lambda))$ is dense in the Hilbert space $\mathcal{D}(S_{\lambda''}) = \mathcal{D}(T^{-1} \cdot T')$, it follows that $K'\mu(\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))$ is dense in the Hilbert space $\mathcal{D}(K \cdot K'^{-1})$. Thus the pair (μ, K') satisfies all of the conditions in Proposition 3.1. It is clear that $\mathcal{D}(\mu) = U^*\mathcal{D}(\lambda)U$ and $\lambda(Y) = U(K'\mu)(UYU^*)$ for each $Y \in \mathcal{D}(\lambda)$. This completes the proof. □

By Propositions 3.1, 3.3 we have the following

Theorem 3.4. *Let $\mathcal{M}$ be a closed semifinite O*-algebra on $\mathcal{D}$ in $\mathcal{H}$ such that $\mathcal{M}'_{\rm w}\mathcal{D} \subset \mathcal{D}$, and let λ be a generalized vector for $\mathcal{M}$. The following statements are equivalent:*

- (i) *$(\mathcal{M}, \lambda, \lambda')$ is a standard system such that $\overline{Y} \in L^2(\tau'')$ for each $Y \in \mathcal{D}(\lambda)$, where τ'' is a faithful normal semifinite trace on $(\mathcal{M}'_{\rm w})'_+$.*
- (ii) *There exist a closed O*-algebra $\mathcal{N}$ on $\mathcal{E}$ in $\mathcal{K}$, a standard tracial generalized vector μ for $\mathcal{N}$ and a non-singular positive self-adjoint operator K' in $\mathcal{K}$ affiliated with $\mathcal{N}'_{\rm w}$ such that*
- (ii)_1 *$\mu(\mathcal{D}(\mu)) \subset \mathcal{D}(K')$ and $K'\mu((\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))^2)$ is total in $\mathcal{K}$;*
- (ii)_2 *$K'\mu(\mathcal{D}(\mu)^\dagger \cap \mathcal{D}(\mu))$ is dense in the Hilbert space $\mathcal{D}(K \cdot K'^{-1})$, where $K \equiv J_\mu K' J_\mu$;*
- (ii)_3 *$K^{it}\mathcal{E} \subset \mathcal{E}$ and $K^{it}YK^{-it}\lceil\mathcal{E} \in \mathcal{D}(\mu)$ for all $Y \in \mathcal{D}(\mu)$ and $t \in \mathbb{R}$;*
- (ii)_4 *λ is unitarily equivalent to the generalized vector $K'\mu$; that is, there exists a unitary operator U of $\mathcal{K}$ onto $\mathcal{H}$ such that $U^*\mathcal{M}U = \mathcal{N}, U^*\mathcal{D}(\lambda)U = \mathcal{D}(\mu)$ and $\lambda(Y) = U(K'\mu)(U^*YU)$ for each $Y \in \mathcal{D}(\lambda)$.*

Corollary 3.5. *Let $\mathcal{M}$ be an integrable O^*-algebra on $\mathcal{D}$ in $\mathcal{H}$ with a standard tracial vector ξ_0, and let $\xi \in \mathcal{D}$. Then ξ is a standard vector for $\mathcal{M}$ if and only if there exists a non-singular positive self-adjoint operator K' in $\mathcal{H}$ affiliated with $\mathcal{M}'_{\mathrm{w}}$ such that* (a) *$\xi_0 \in \mathcal{D}(K')$ and $K'\xi_0 \in \mathcal{D}$;* (b) *$\mathcal{M}\xi_0$ is dense in the Hilbert space $\mathcal{D}(K \cdot K'^{-1})$, where $K \equiv J_{\xi_0} K' J_{\xi_0}$;* (c) *$K^{it} \lceil \mathcal{D} \in \mathcal{M}$ for each $t \in \mathbb{R}$;* (d) *ξ is unitarily equivalent to $K'\xi_0$.*

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