PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 11, November 1997, Pages 3257–3258
S 0002-9939(97)03963-4

HARMONIC POLYNOMIALS AND THE DIVISIBILITY PROBLEM

V. V. KARACHIK

(Communicated by J. Marshall Ash)

ABSTRACT. An easy way to construct a fist harmonic polynomial component of any polynomial is given.

If we tried to divide any polynomial $P(x)$ ($x \in R^n$) by the polynomial $L(x)$ using the ordinary Euclidean algorithm then for $n > 1$ we would meet the problem: What should be a residue? We would not find a reasonable answer if we were keeping in mind that the degree of a residue must be less than the degree of $L(x)$. Nevertheless we may define a "division" of the polynomial $P(x)$ by the polynomial $L(x)$ by the equality $P(x) = Q(x)L(x) + H(x)$, where the residue $H(x)$ is determined not as a polynomial of degree less than degree of $L(x)$ but as a polynomial solution of the equation $L(D)u(x) = 0$. Here operator $L(D)$ is obtained from the polynomial $L(x)$ by replacing each variable x_i on the differential operator $\partial/\partial x_i$. In this case for each polynomial $P(x)$ there exist the only polynomials $Q(x)$ and $H(x)$ such that the equality $P(x) = Q(x)L(x)+H(x)$ holds under the condition $L(D)H(x) = 0$ [1]. If $L(x) = |x|^2 \equiv x_1^2 + \cdots + x_n^2$ this fact was proved in [2]. The proof's method of the above statement does not permit us to construct the polynomial $H(x)$ by the polynomial $P(x)$. In the general case it is not a simple problem. Let us consider an easy way to find the polynomial $H(x)$ for the special form of the polynomial $L(x)$, i.e., if $L(x) = |x|^2$ and $n > 2$.

Let $L(D)$ be the Laplace operator, i.e., $L(D) = \Delta$.

Lemma. *Let $H_m(x)$ be a homogeneous harmonic polynomial of m-th degree, $H_m^*(x)$ be the Kelvin transformation of $H_m(x)$ ($H_m^*(x) \equiv |x|^{2-n} H_m(x/|x|^2)$) and $(k, 2)_m = k(k+2)\ldots(k+2m-2)$. If $n > 2$ and $x \neq 0$ then the following formula holds:*

$$H_m^*(x) = \frac{(-1)^m}{(n-2,2)_m} H_m(D)|x|^{2-n}. \tag{1}$$

Proof. We shall employ the induction on m. Set $m = 1$. Then for $k = 1, \ldots, n$ we can easily write the equality

$$\frac{\partial}{\partial x_k} \frac{|x|^{2-n}}{2-n} = \frac{1}{|x|^{n-2}} \frac{x_k}{|x|^2} \equiv x_k^*,$$

and therefore for $m = 1$ Eq.(1) is true.

Received by the editors June 9, 1995 and, in revised form, May 9, 1996.
1991 *Mathematics Subject Classification.* Primary 31A05; Secondary 11C08.
Key words and phrases. Harmonic polynomials, divisibility.

Suppose that for $m < k$ the lemma is true and prove it for $m = k$. Let us use the Euler formula for homogeneous functions. We get

$$H_k(x) = \sum_{i=1}^{n} \frac{x_i}{k} H_k^{(i)}(x), \tag{2}$$

where $H_k^{(i)}(x) = \partial/\partial x_i H_k(x)$. Obviously the polynomials $H_k^{(i)}(x)$ are harmonic polynomials of degree $k-1$. Making use of Eq.(2), by the induction hypotheses we can write

$$H_k(D)|x|^{2-n} = \sum_{i=1}^{n} \frac{1}{k} \frac{\partial}{\partial x_i} H_k^{(i)}(D)|x|^{2-n}$$

$$= \sum_{i=1}^{n} \frac{(-1)^{k-1}}{k} (n-2,2)_{k-1} \frac{\partial}{\partial x_i} \left(H_k^{(i)}\right)^* (x).$$

Keeping in mind that

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(H_k^{(i)}\right)^* (x) = -(2k+n-4) \sum_{i=1}^{n} \left(x_i H_k^{(i)}\right)^* (x),$$

we get

$$H_k(D)|x|^{2-n} = (-1)^k (n-2,2)_{k-1}(2k+n-4) \left(\sum_{i=1}^{n} \frac{x_i}{k} H_k^{(i)}\right)^* (x).$$

Again, making use of Eq.(2) and observing that $(n-2,2)_{k-1}(2k+n-4) = (n-2,2)_k$ we get Eq.(1), and the proof is complete. □

Let $P(x)$ be an arbitrary polynomial. Represent it in the form $P(x) = \sum_m P_m(x)$.

Theorem. *Suppose that $P(x) = Q(x)|x|^2 + H(x)$ and $H(x)$ is a harmonic polynomial. Then $H(x)$ can be found from the equality*

$$H(x) = \sum_m (-1)^m \frac{|x|^{2m+n-2}}{(n-2,2)_m} P_m(D)|x|^{2-n}.$$

Proof. Since $P(x) = Q(x)|x|^2 + H(x)$, then for $x \neq 0$ we get

$$P_m(D)|x|^{2-n} = H_m(D)|x|^{2-n}.$$

If we take advantage of the lemma then we easily get the desired result. □

References

[1] V. V. Karachik, O polinomialnyh reshenijah sistem linejnyh differenzialnyh uravnenij. Voprosi Vychislitelnoy i prikladnoy matematiki. v.82, Tashkent, 1987, s.41-48. MR **91f:**34008

[2] E. M. Stein, G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton, New Jersey, Princeton University Press, 1971. MR **46:**4102

Institute of Cybernetics of the Academy of Science of Uzbekistan, 34, F. Hodzhaev St., Tashkent, 700143, Uzbekistan

E-mail address: `karachik@uwed.freenet.uz`