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A CHARACTERIZATION OF THE LEINERT PROPERTY

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Abstract. Let G be a discrete group and denote by λ_G its left regular representation on $\ell_2(G)$. Denote further by $\mathbf{F}_n$ the free group on n generators $\{g_1, g_2, \dots, g_n\}$ and λ its left regular representation. In this paper we show that a subset $S = \{t_1, t_2, \dots, t_n\}$ of G has the Leinert property if and only if for some real positive coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ the identity

$$\left\|\sum_{i=1}^n \alpha_i \, \lambda_G(t_i)\right\|_{C^*_\lambda(G)} = \left\|\sum_{i=1}^n \alpha_i \, \lambda(g_i)\right\|_{C^*_\lambda(\mathbf{F}_n)}$$

holds. Using the same method we obtain some metric estimates about abstract unitaries $U_1, U_2, \dots, U_n$ satisfying the similar identity $\left\|\sum_{i=1}^n U_i \otimes \overline{U_i}\right\|_{\min} = 2\sqrt{n-1}$.

Notation. Throughout in this paper G will denote a discrete group with unit element e and $C^*_\lambda(G)$ the sub-C^*-algebra of $B(\ell_2(G))$ generated by its left regular representation λ_G. This algebra is equipped with the trace state $\tau_G(X) = \langle X\delta_e, \delta_e\rangle$. A subset $\{t_1, t_2, \dots, t_n\}$ of G is called *free* if it generates a copy of $\mathbf{F}_n$, the free group on n generators. We shall denote the canonical generators of the free group by $\{g_1, g_2, \dots, g_n\}$. When considering the free group we shall omit the subscript in $\lambda_{\mathbf{F}_n}$. We shall almost exclusively deal with finitely generated groups and repeatedly use the fact that given a generating set $\{h_1, h_2, \dots, h_n\}$ for the group G there is a (unique) quotient mapping $q : \mathbf{F}_n \to G$ with $q(g_i) = h_i \quad \forall i \in \{1, 2, \dots, n\}$.

The following proposition collects some results from [K, Lemma 3.1 and Theorem 3]:

Proposition 1. *Let $\{h_1, h_2, \dots, h_n\}$ be a generating set of the group G and denote by $\alpha_1, \alpha_2, \dots, \alpha_n$ positive real numbers. Then*

$$\left\|\sum_{i=1}^n \alpha_i \left(\lambda_G(h_i) + \lambda_G(h_i)^*\right)\right\|_{C^*_\lambda(G)} \geq \left\|\sum_{i=1}^n \alpha_i \left(\lambda(g_i) + \lambda(g_i)^*\right)\right\|_{C^*_\lambda(\mathbf{F}_n)}.$$

Moreover in the case of equal coefficients,

$$\left\|\sum_{i=1}^n (\lambda_G(h_i) + \lambda_G(h_i)^*)\right\|_{C^*_\lambda(G)} = \left\|\sum_{i=1}^n (\lambda(g_i) + \lambda(g_i)^*)\right\|_{C^*_\lambda(\mathbf{F}_n)} = 2\sqrt{2n-1}$$

if and only if $\{h_1, h_2, \dots, h_n\}$ is a free set.

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This result has a graph-theoretical interpretation, where the operator in consideration corresponds to the combinatorial Laplacian on the Cayley graph of the group G; see [Pa] in particular for a generalization of the second statement. However it remained unclear whether the free group can be characterized by the norms of non-selfadjoint operators, and this question will be the subject of this paper. The following proposition is an extension of [K, Lemma 3.1].

Proposition 2. *Let G and H be discrete groups and $q : G \to H$ a group homomorphism. Then for every finite subset $\{t_1, t_2, \dots, t_m\}$ of G and corresponding nonnegative real numbers $\alpha_1, \alpha_2, \dots, \alpha_m$ the inequality*

$$\left\| \sum_{i=1}^{m} \alpha_i \, \lambda_H(q(t_i)) \right\|_{C^*_\lambda(H)} \geq \left\| \sum_{i=1}^{m} \alpha_i \, \lambda_G(t_i) \right\|_{C^*_\lambda(G)} \tag{1}$$

holds.

Proof. We will use the following well known fact about the noncommutative L^p-norms associated to τ_G:

$$\|X\|_{C^*_\lambda(G)} = \sup_{1 \leq p < \infty} (\tau_G((X^*X)^p))^{1/2p}.$$

It suffices to consider the integer values of p. Denote by $W_p^{alt}(A)$ the set of all alternating words of length $2p$ in the letters $A = \{t_1, t_2, \dots, t_m\}$:

$$W_p^{alt}(A) = \left\{ t_{i_1}^{-1} t_{j_1} t_{i_2}^{-1} t_{j_2} \cdots t_{i_p}^{-1} t_{j_p} : i_1, j_1, i_2, j_2, \dots, i_p, j_p \in \{1, 2, \dots, m\} \right\}.$$

For $v = t_{i_1}^{-1} t_{j_1} t_{i_2}^{-1} t_{j_2} \cdots t_{i_p}^{-1} t_{j_p} \in W_p^{alt}(A)$ we use the following abbreviation:

$$\alpha_v = \alpha_{i_1} \alpha_{j_1} \alpha_{i_2} \alpha_{j_2} \cdots \alpha_{i_p} \alpha_{j_p}. \tag{2}$$

Then we can write (note that we identify words in $W_p^{alt}(A)$ with the corresponding elements in the group G):

$$\begin{aligned} \left\| \sum_{i=1}^{m} \alpha_i \, \lambda_H(q(t_i)) \right\|_{C^*_\lambda(H)} &= \sup_p \left(\tau_H \left(\sum_{i,j=1}^{m} \alpha_i \alpha_j \, \lambda_H(q(t_i^{-1} t_j)) \right)^p \right)^{1/2p} \\ &= \sup_p \left(\sum_{v \in W_p^{alt}(A)} \alpha_v \, \tau_H(\lambda_H(q(v))) \right)^{1/2p} = \sup_p \left(\sum_{\substack{v \in W_p^{alt}(A) \\ q(v)=e}} \alpha_v \right)^{1/2p} \\ &\geq \sup_p \left(\sum_{\substack{v \in W_p^{alt}(A) \\ v=e}} \alpha_v \right)^{1/2p} = \left\| \sum_{i=1}^{m} \alpha_i \, \lambda_G(t_i) \right\|_{C^*_\lambda(G)}. \end{aligned}$$

□

In [P2] the following more general statement is proved:

Proposition 3. *Let H be a Hilbert space and $U_1, U_2, \dots, U_n$ a finite sequence of unitary operators acting on H. Consider the representation π of $\mathbf{F}_n$ which is determined by the condition $\pi(g_i) = U_i$ for all $i \in \{1, 2, \dots, n\}$. Then for every set of words $\{w_1, w_2, \dots, w_m\} \in \mathbf{F}_n$ and every sequence $\alpha_1, \alpha_2, \dots, \alpha_m$ of positive real numbers we have*

$$\Big\| \sum_{i=1}^{m} \alpha_i \, \pi(w_i) \otimes \overline{\pi(w_i)} \Big\|_{\min} \geq \quad \Big\| \sum_{i=1}^{m} \alpha_i \, \lambda(w_i) \Big\|_{C^*_\lambda(\mathbf{F}_n)}.$$

The set $\{g_1, g_1^{-1}, g_2, g_2^{-1}, \dots, g_n, g_n^{-1}\}$ is only a special case of a *Leinert set*, a concept which appeared first in [Le].

Definition 4 ([A-O, Definition III B and Theorem III D]).
A subset $A = \{t_1, t_2, \dots, t_n\}$ of a discrete group G with unit element e is called a *Leinert set* if it satisfies one of the following equivalent conditions:

- Every sequence $t_{i_1}, t_{j_1}, t_{i_2}, t_{j_2}, \dots, t_{i_m}, t_{j_m}$ with i_k, $j_k \in \{1, 2, \dots, n\}$ such that $i_1 \neq j_1 \neq i_2 \neq j_2 \neq \dots \neq i_m \neq j_m$ satisfies
$$t_{i_1}^{-1} t_{j_1} t_{i_2}^{-1} t_{j_2} \cdots t_{i_m}^{-1} t_{j_m} \neq e.$$
- The set A can be written as $y(B \cup \{e\})$, where B is a free subset of G and $y \in G$.

The next proposition extends Kesten's formula for the norm (see also [B] for a weaker version). Simplified proofs can be found in [P-P] and [W]. For recent examples of Leinert sets in one-relator groups see [C-V].

Proposition 5 ([A-O, Theorem IV J]). *Let $n \geq 2$ and $\{t_1, \dots, t_n\}$ be a Leinert set in a discrete group G. Then*

$$\Big\| \sum_{i=1}^{n} \lambda_G(t_i) \Big\|_{C^*_\lambda(G)} = 2\sqrt{n-1}.$$

The next proposition is standard.

Proposition 6. *If $u_1, u_2, \dots, u_n$ are unitaries in a C^*-algebra A for which there exist positive real numbers $\beta_1, \beta_2, \dots, \beta_n$ satisfying*

$$\Big\| \sum_{i=1}^{n} \beta_i \, u_i \Big\| = \sum_{i=1}^{n} \beta_i$$

then there exists a state φ of A such that for every $m \in \mathbf{N}$

$$\varphi(u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_m}^* u_{j_m}) = 1 \qquad \forall i_k, j_k = 1, 2, \dots, n.$$

In particular, if $t_1, t_2, \dots, t_n$ are some elements of the group G then the subgroup of G generated by $\{t_i^{-1} t_j : i, j = 1, 2, \dots, n\}$ is amenable if and only if $\| \sum \beta_i \, \lambda_G(t_i) \| = \sum \beta_i$ for some positive coefficients $\beta_1, \beta_2, \dots, \beta_n$.

Proof. By a compactness argument, there is a state on A such that $\varphi(\sum \beta_i \beta_j \, u_i^* u_j) = \| \sum \beta_i \beta_j \, u_i^* u_j \| = \sum \beta_i \beta_j$ It follows that $\varphi(u_i^* u_j) = 1$ for all $i, j = 1, 2, \dots, n$ and by induction on m we have $\phi(u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_m}^* u_{j_m}) = 1$ for any choice of indices i_k, j_k. Indeed,

$$\big| \phi(u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_m}^* u_{j_m} - u_i^* u_j u_{i_1}^* u_{j_1} u_{i_2}^* u_{j_2} \cdots u_{i_m}^* u_{j_m}) \big| \leq \phi((I - u_i^* u_j)^* (I - u_i^* u_j))^{1/2} = 0.$$

Now in the case where $u_i = \lambda_G(t_i)$ this implies that the trivial representation of the group generated by $t_i^{-1}t_j$ extends to a representation of the reduced group C^*-algebra, which implies that it is amenable. □

The following lemma and its corollary will play a key rôle in the generalization of Kesten's result.

Lemma 7 ([H-R-V1, Lemma 8]). *Let μ be a probability measure on $\mathbf{R}$ with compact support and suppose*

$$-m = \min \operatorname{supp} \mu < \max \operatorname{supp} \mu = M,$$
$$\mu_n = \int_{\mathbf{R}} t^n \, d\mu(t) \geq 0 \text{ for all } n \in \mathbf{N}.$$

Then

$$\limsup_{n\to\infty} (\mu_n)^{1/n} = M = \max(m, M).$$

Corollary 8 ([K, Lemma 3.2]). *For every subset $\{t_1, t_2, \dots, t_n\}$ of G and every sequence of positive real numbers $\alpha_0, \alpha_1, \dots, \alpha_n$ we have*

$$\left\|\alpha_0 I + \sum_{i=1}^n \alpha_i \left(\lambda_G(t_i) + \lambda_G(t_i)^*\right)\right\| \quad = \quad \alpha_0 + \left\|\sum_{i=1}^n \alpha_i \left(\lambda_G(t_i) + \lambda_G(t_i)^*\right)\right\|$$

and a similar result holds for general unitaries:

$$\left\|\alpha_0 I + \sum_{i=1}^n \alpha_i \left(U_i \otimes \overline{U_i} + U_i^* \otimes \overline{U_i^*}\right)\right\| \quad = \quad \alpha_0 + \left\|\sum_{i=1}^n \alpha_i \left(U_i \otimes \overline{U_i} + U_i^* \otimes \overline{U_i^*}\right)\right\|.$$

Proof. This follows from Lemma 7 applied to the probability measure μ on the spectrum of the self-adjoint operator

$$T = \sum_{i=1}^n \alpha_i \left(\lambda(t_i) + \lambda(t_i)^*\right)$$

which is determined by the moments

$$\mu_n = \tau_G(T^n).$$

For the case of general unitaries $\hat{U}_i = U_i \otimes \overline{U_i}$ we refer to [P1, Example 5.6] where the following formula is proved: For any finite sequence of operators $x_1, x_2, \dots, x_n$ on some Hilbert space H we have

$$\left\|\sum x_i \otimes \overline{x_i}\right\|_{\min} = \sup_{y,z \in (S_2^+)_1} \sum \operatorname{tr}\left(x_i y x_i^* z\right) \tag{3}$$

where $(S_2^+)_1$ is the intersection of the unit ball of the Hilbert-Schmidt class operators on H with the cone of positive operators. In our case the operator is self-adjoint and we can restrict (3) to the symmetric part:

$$\begin{aligned}\left\|\alpha_0 \, I + \sum_{i=1}^n \alpha_i \left(\hat{U}_i + \hat{U}_i^*\right)\right\|_{\min} &= \sup_{y \in (S_2^+)_1} \operatorname{tr} \left(\alpha_0 \, y^2 + 2 \sum_{i=1}^n \hat{U}_i y \hat{U}_i^* y\right) \\ &= \alpha_0 + \left\|\sum_{i=1}^n \alpha_i \left(\hat{U}_i + \hat{U}_i^*\right)\right\|_{\min}.\end{aligned}$$

□

Now we are ready to prove the main result.

Theorem 9. *Let G be a discrete group, $n \geq 3$ an integer and let $t_1, t_2, \dots, t_n$ be some elements in G. Then for any sequence $\alpha_1, \alpha_2, \dots, \alpha_n$ of strictly positive numbers we have*

$$\Big\| \sum_{i=1}^{n} \alpha_i \, \lambda_G(t_i) \Big\|_{C^*_\lambda(G)} = \Big\| \sum_{i=1}^{n} \alpha_i \, \lambda(g_i) \Big\|_{C^*_\lambda(\mathbf{F}_n)}$$

if and only if $\{t_1, t_2, \dots, t_n\}$ is a Leinert set. In particular, this is equivalent to the identity

$$\Big\| \sum_{i=1}^{n} \lambda_G(t_i) \Big\|_{C^*_\lambda(G)} = 2\sqrt{n-1}.$$

Proof. We only have to show the "only if" part and we shall use the method introduced in [K, Theorem 3] (see also [B], where similar methods are used. We are grateful to A. Valette for bringing this article to our attention). Let us compare the norms of the operators

$$\tilde{T} = \sum_{i=1}^{n} \alpha_i \, \lambda_G(t_i) \qquad \text{and} \qquad T = \sum_{i=1}^{n} \alpha_i \, \lambda(g_i).$$

We can assume that G is generated by the subset $\{t_1, t_2, \dots, t_n\}$. Suppose that the set $\{t_1, t_2, \dots, t_n\}$ does not have the Leinert property. This means that there are an integer m and a sequence of indices $i_1 \neq j_1 \neq i_2 \neq j_2 \neq \dots \neq i_m \neq j_m$, with $i_k, j_k \in \{1, 2, \dots, n\}$ and such that

$$t_{i_1}^{-1} t_{j_1} t_{i_2}^{-1} t_{j_2} \cdots t_{i_m}^{-1} t_{j_m} = e.$$

Let $g_1, g_2, \dots, g_n$ be the generators of the free group $\mathbf{F}_n$ and let $q : \mathbf{F}_n \to G$ be the quotient mapping associated to the generating set $\{t_1, t_2, \dots, t_n\}$. Taking the same indices as above, set

$$w := g_{i_1}^{-1} g_{j_1} g_{i_2}^{-1} g_{j_2} \cdots g_{i_m}^{-1} g_{j_m}.$$

Since $\{g_1, g_2, \dots, g_n\}$ *is* a Leinert set, the word w does not reduce to the identity in $\mathbf{F}_n$ but it is in the kernel of q. We can assume without loss of generality that w is cyclically reduced, i.e. $j_m \neq i_1$. For if this is not the case, we can consider the word

$$g_{i_m}^{-1} g_{i_1} w \, g_{i_1}^{-1} g_{i_m} = g_{i_m}^{-1} g_{j_1} g_{i_2}^{-1} g_{j_2} \cdots g_{i_{m-1}}^{-1} g_{j_{m-1}}$$

which is cyclically reduced and still in the kernel of q. Note that it cannot be reduced to the empty word in this way, since the conjugacy class of the latter in $\mathbf{F}_n$ is trivial.

So we can assume for the sake of clarity that $i_1 = 1$ and $j_m = 2$. Now consider the elements

$$w' = g_1^{-1} g_3 \, w \, g_3^{-1} g_1,$$
$$w'' = g_2^{-1} g_3 \, w \, g_3^{-1} g_2$$

which are both in the kernel of q and have infinite order, because w has, being cyclically reduced. Moreover, they form a free set, since there are no possible cancellations in non-trivial reduced words built out of them. Thus we can apply Proposition 6 to the operator

$$T_1 = \alpha_{w'}\left(\lambda(w') + \lambda(w')^*\right) + \alpha_{w''}\left(\lambda(w'') + \lambda(w'')^*\right)$$

and we get

$$\|T_1\| < 2(\alpha_{w'} + \alpha_{w''}) \tag{4}$$

because the group generated by $\{{w'}^{-1}w'', {w'}^2, {w''}^2\}$ is not amenable.

To compare the norms of T and $\tilde{T}$ we use the identity

$$\|X\|^2 = \|X^*X\| = \|(X^*X)^p\|^{1/p},$$

which is a consequence of Gel′fand's theorem. Thus it suffices to show that for some p

$$\|(T^*T)^p\| < \|(\tilde{T}^*\tilde{T})^p\|.$$

We consider $p = m+2$ so that $2p = |w'| = |w''|$. Denote by $W_p^{alt}(n)$ the set of all alternating words of length $2p$ in n generators:

$$W_p^{alt}(n) = \{g_{i_1}^{-1}g_{j_1}g_{i_2}^{-1}g_{j_2}\cdots g_{i_p}^{-1}g_{j_p} : i_1, j_1, i_2, j_2, \ldots, i_p, j_p \in \{1, 2, \ldots, n\}\}.$$

Using the notation of (2) we can write

$$\begin{aligned}(T^*T)^{m+2} &= \sum_{v \in W_{m+2}^{alt}(n)} \alpha_v \lambda(v)\\ &= T_0 + T_1\end{aligned}$$

where

$$T_0 = \sum_{v \in W_{m+2}^{alt}(n) \setminus \{w', {w'}^{-1}, w'', {w''}^{-1}\}} \alpha_v \lambda(v)$$

and T_1 is the same as above. Obviously, setting

$$\tilde{T}_0 = \sum_{v \in W_{m+2}^{alt}(n) \setminus \{w', {w'}^{-1}, w'', {w''}^{-1}\}} \alpha_v \lambda_G(q(v))$$

we have

$$(\tilde{T}^*\tilde{T})^{m+2} = \tilde{T}_0 + 2(\alpha_{w'} + \alpha_{w''})I.$$

Observe that T_0 (and hence $\tilde{T}_0$) is selfadjoint with positive coefficients so that we can apply (1) and Corollary 8 and we get

$$\begin{aligned}\|(T^*T)^{m+2}\| \le \|T_0\| + \|T_1\| &< \|\tilde{T}_0\| + 2(\alpha_{w'} + \alpha_{w''})\\ &= \|\tilde{T}_0 + 2(\alpha_{w'} + \alpha_{w''})I\| = \|(\tilde{T}^*\tilde{T})^{m+2}\|.\end{aligned}$$

□

Remark. Actually in the case of equal coefficients the above proof gives the following quantitative estimate (cf. [K, (4.15)] and [Pa, Theorem 3.2]): With the notation as in the proof, let m denote the minimal length of a nontrivial word in the kernel of the quotient mapping q. Then

$$\|\tilde{T}\| \ge \|T\| + \frac{2(n-1) - 2\sqrt{2n-3}}{(2m+4)\, n^{2m+3}}.$$

Indeed, when there are n free generators, we can build a free set $\{w_1, w_2, \dots, w_{n-1}\}$ in $\mathbf{F}_n$ by setting $w_i = g_i^{-1} g_n\, w\, g_n^{-1} g_i$ for $i \in \{1, 2, \dots, n-1\}$. Then

$$\left\| \sum_{i=1}^{n-1} (\lambda(w_i) + \lambda(w_i)^*) \right\| = 2\sqrt{2n-3}$$

and by modifying T_0 and T_1 (resp. $\tilde{T}_0$ and $\tilde{T}_1$ appropriately), we have the inequality

$$\|\tilde{T}\|^{2(m+2)} \geq \|T\|^{2(m+2)} + 2(n-1) - 2\sqrt{2n-3}.$$

Now a simple convexity argument and the fact that $\|\tilde{T}\| \leq n$ yield the claim.

The following corollary gives a characterization of the free group in terms of the norms operators in the reduced C^*-algebra. Note that the free group cannot be characterized by the spectrum of the sum of the generators, as shown in [H-R-V, sect.4].

Corollary 10. *Let G be a discrete group and $S = \{t_0 = e, t_1, \dots, t_n\}$ be a generating subset with $n \geq 2$. Then*

$$2\sqrt{n} \leq \left\| \sum_{i=0}^{n} \lambda_G(t_i) \right\|$$

with equality if and only if G is the free group on n generators.

A similar proof yields the following result about unitaries which do not necessarily arise from a regular representation of a discrete group, as for example in [L-P-S].

Theorem 11. *Let $U_1, \dots, U_n$ be unitary operators acting on some Hilbert space H and suppose*

$$\left\| \sum_{i=1}^{n} U_i \otimes \overline{U_i} \right\|_{\min} = 2\sqrt{n-1}.$$

Then for all products of the form $V = U_{i_1}^{-1} U_{j_1} U_{i_2}^{-1} U_{j_2} \cdots U_{i_m}^{-1} U_{j_m}$ with $i_1 \neq j_1 \neq i_2 \neq j_2 \neq \dots \neq i_m \neq j_m$ we have

$$\|V \otimes \overline{V} - I\| \geq 1.$$

Proof. To simplify notation we use the unitary representation

$$\hat{\pi} = \pi \otimes \overline{\pi} : \mathbf{F}_n \to B(H \otimes_2 \overline{H})$$

which we already introduced in Proposition 3 and which maps the generators g_i to the corresponding unitaries $\hat{U}_i = U_i \otimes \overline{U_i}$. Now let

$$v_0 = g_{i_1}^{-1} g_{j_1} g_{i_2}^{-1} g_{j_2} \cdots g_{i_m}^{-1} g_{j_m}$$

be a word as in the claim and set $\hat{V} = \hat{\pi}(v_0) = V \otimes \overline{V}$. We can assume v_0 to be cyclically reduced, because $\|\hat{V} - I\| = \|W\hat{V}W^{-1} - I\|$ for any unitary W and in the case v_0 is not cyclically reduced, we can take a cyclically reduced one in its conjugacy class. We can of course assume that $i_1 = 1$ and $j_m = 2$. Now for the cyclically reduced word v_0 the set

$$F_k = \{(g_1^{-1} g_3)^r\, v_0\, (g_1^{-1} g_3)^{-r} : r = 1, 2, \dots, k\}$$

is a free set and we can do the same estimate as in the above proof. Setting $T = \sum_{i=1}^{n} \lambda(g_i)$ as above and $\hat{T} = \sum_{i=1}^{n} \hat{U}_i$ we can identify F_k with some subset of the set of unreduced words $W^{alt}_{m+2k}(n)$, e.g. the set $\{(g_1^{-1}g_3)^r v_0 (g_1^{-1}g_3)^{-r}(g_1^{-1}g_1)^{k-r} \colon\ r = 1, 2, \dots, k\}$, and get

$$\begin{aligned}\|(T^*T)^{m+2k}\| &= \Big\| \sum_{v\in W^{alt}_{m+2k}(n)} \lambda(v) \Big\| \\ &\le \Big\| \sum_{v\in W^{alt}_{m+2k}(n)\setminus(F_k\cup F_k^{-1})} \lambda(v)\Big\| + \Big\| \sum_{v\in F_k} \lambda(v)+\lambda(v)^*\Big\| \\ &\le \Big\| \sum_{v\in W^{alt}_{m+2k}(n)\setminus(F_k\cup F_k^{-1})} \hat\pi(v)\Big\| + 2\sqrt{2k-1}.\end{aligned}$$

The last inequality follows from (1) and Proposition 5. We can now apply Corollary 8 to the first term on the right-hand side and we obtain

$$\begin{aligned}\|(T^*T)^{m+2k}\| &\le \Big\| \sum_{v\in W^{alt}_{m+2k}(n)\setminus(F_k\cup F_k^{-1})} \hat\pi(v) + \sum_{v\in F_k} 2I \Big\| + 2\sqrt{2k-1} - 2k \\ &\le \Big\| \sum_{v\in W^{alt}_{m+2k}(n)} \hat\pi(v)\Big\| + \Big\| \sum_{v\in F_k} 2I - \hat\pi(v) - \hat\pi(v)^*\Big\| + 2\sqrt{2k-1} - 2k \\ &\le \|(\hat T^*\hat T)^{m+2k}\| + 2k\|\hat\pi(v_0) - I\| + 2\sqrt{2k-1} - 2k.\end{aligned}$$

This together with the assumption $\|T\| = \|\hat T\|$ yields

$$\|\hat V - I\| \ge 1 - \frac{\sqrt{2k-1}}{k}$$

and since this holds for all k, the proof is complete. □

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Note added in proof

We noticed a connection to the paper of V. Flory, "Estimating norms in C^*-algebras of discrete groups", Math. Ann. **224** (1976) 41–52. Using our results Theorem 8 in the latter paper can be sharpened for finite sets as follows.

Let G be a discrete group and $E \subset G$ a finite subset. Theorem 8 can be sharpened for finite subsets as follows. Define for $K \subset G$ the *Leptin constant* $\omega(K) = \inf_{\varnothing\ne U\subset G \text{ finite}} \frac{\#(KU)}{\#(U)}$. Then the following are equivalent.

(1) E has the Leinert property.
(2) For every $y \in E$ the set $\{y^{-1}x : x \in E \text{ and } x \ne y\}$ is a free subset of G.
(3) $\omega(E) = \#(E) - 1$.

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