

LEMPERT MAPPINGS AND SYMPLECTIC FORMS

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ABSTRACT. We use Lempert's version of Riemann mapping to construct non-equivalent symplectic forms on an ellipsoid in \mathbf{C}^n .

Any bounded pseudoconvex domain $D \subseteq \mathbf{C}^n$ admits a strictly plurisubharmonic *exhausting* function u . If u is smooth (or at least C^2) then $\omega = dd^c u = 2i\partial\bar{\partial}u$ is a symplectic form on D . The following theorem was proved in [2].

Theorem A (Eliashberg, Gromov). *Let u_1, u_2 be two smooth strictly plurisubharmonic exhausting functions on a bounded strictly pseudoconvex domain D :*

$$D = \{u_1 < \infty\} = \{u_2 < \infty\}.$$

Then the forms $\omega_1 = dd^c u_1$ and $\omega_2 = dd^c u_2$ are symplectomorphic, i.e. there exists a diffeomorphism $\psi : D \rightarrow D$ such that $\psi^ \omega_2 = \omega_1$.*

As $u_i(z) \rightarrow \infty$ for $z \rightarrow \partial D$ it can easily be seen that the volume of D with respect to the volume forms $\Omega_i = \omega_i \wedge \cdots \wedge \omega_i$ ($i = 1, 2$) is infinite. We will construct an example which shows that Theorem A no longer holds true for the finite volume case, i.e. when we give the domain not by *exhausting* but by *defining* functions.

Let D be a bounded strictly pseudoconvex domain represented by a strictly plurisubharmonic smooth *defining* function u_1 , i.e. $D = \{u_1(z) < 0\} \subseteq \mathbf{C}^n$ with $du_1|_{\partial D} \neq 0$. D is a symplectic manifold with the 2-form $\omega_1 = dd^c u_1$. Observe that the volume of D with respect to the volume form $\Omega_1 = \omega_1 \wedge \cdots \wedge \omega_1$ is finite.

If u_2 is another defining function for D that is equal to u_1 in a neighbourhood of the boundary ∂D then $\int_D \Omega_1 = \int_D \Omega_2$ as follows from Stokes' theorem. Moreover, using Moser's deformation argument (see e.g. [1], p.20) we can easily construct a symplectomorphism between (D, ω_1) and (D, ω_2) . If u_1 and u_2 are not equal in a neighbourhood of ∂D the volume condition alone is not sufficient to guarantee the existence of a symplectomorphism as will be shown in the following example.

The basis of our construction is Lempert's theorem stating that for any strictly convex domain $D \subseteq \mathbf{C}^n$ there is a homeomorphism of the unit ball $\rho : B \rightarrow D$ which has many similar properties to the usual Riemann mapping in \mathbf{C} . The property we need here is that the function $u : D \rightarrow \mathbf{R}$, $u(z) = \frac{1}{4}(|\rho^{-1}(z)|^2 - 1)$ is a plurisubharmonic defining function of D and moreover the relation $dd^c u = \rho_*^{-1} \omega_1$ holds where $\omega_1 = \sum dy_i \wedge dx_i = \frac{1}{4} dd^c (|z|^2 - 1)$ is the standard symplectic form in \mathbf{C}^n (see e.g. [5], Lemma 2.6).

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We are going to use furthermore the fundamental theorem of Gromov which is according to A. Weinstein the geometric expression of the uncertainty principle (see e.g. [4], p.358 for explanation). Gromov's theorem states that if there exists a symplectic embedding

$$(B(r), \omega_1) \hookrightarrow (Z_1(R), \omega_1)$$

of the ball in \mathbf{R}^{2n} of radius r into the cylinder of radius R

$$Z_1(R) = \{(x_1, y_1, \dots, x_n, y_n) \in \mathbf{R}^{2n} : x_1^2 + y_1^2 < R^2\}$$

then necessarily $r \leq R$.

Let $A : \mathbf{C}^2 \rightarrow \mathbf{C}^2$, $A(x_1, y_1, x_2, y_2) = (ax_1, ay_1, b_1x_2, b_2y_2)$ be a real linear map and $B = \{|z| < 1\}$ be the unit ball in \mathbf{C}^2 . Consider the real ellipsoid

$$\begin{aligned} E &= A^{-1}(B) = \{z \in \mathbf{C}^2 : |A(z)|^2 - 1 < 0\} \\ &= \{x \in \mathbf{R}^4 : a^2x_1^2 + a^2y_1^2 + b_1^2x_2^2 + b_2^2y_2^2 - 1 < 0\}. \end{aligned}$$

We are going to construct two plurisubharmonic defining functions u_1 and u_2 for E and consider the symplectic forms $\omega_i = dd^c u_i$, $i = 1, 2$. These will have the properties:

- u_i are smooth and $\int_E \omega_1 \wedge \omega_1 = \int_E \omega_2 \wedge \omega_2$;
- there exists no diffeomorphism $\psi : E \rightarrow E$, $\psi^* \omega_2 = \omega_1$.

First we choose the positive numbers a, b_1, b_2 carefully to fulfill the relations

1. $2a^2 = b_1^2 + b_2^2$,
2. $a^2 b_1 b_2 = 1$,
3. $a^2 > 1$ and $b_1 b_2 < 1$.

(For example, choose $a = 2$, $b_1 = \frac{1}{2\sqrt{2}}(\sqrt{15} + \sqrt{17})$, $b_2 = \frac{1}{4b_1}$.)

The function $u_1(z) = \frac{1}{4a^2}(|A(z)|^2 - 1)$ is a plurisubharmonic defining function for E and by relation 1 we have $\omega_1 = dd^c u_1 = \sum_{i=1}^2 dy_i \wedge dx_i$.

On the other hand we consider the Lempert mapping $\rho : B \rightarrow E$, $\rho(0) = 0$, which exists as E is strictly convex, and put $u_2(z) = \frac{1}{4}(|\rho^{-1}(z)|^2 - 1)$. Then u_2 is another plurisubharmonic defining function for E and $\omega_2 = \rho_*^{-1} \omega_1$ where ω_1 is now the standard symplectic structure on the unit ball B . The forms ω_1 and ω_2 give the same volume for E . Indeed by relation 2:

$$\begin{aligned} \text{vol}_{\omega_1}(E) &= \int_E \omega_1 \wedge \omega_1 = \int_B (A^{-1})^*(\omega_1 \wedge \omega_1) = \frac{1}{a^2 b_1 b_2} \int_B \omega_1 \wedge \omega_1 = \int_B \omega_1 \wedge \omega_1, \\ (1) \quad \text{vol}_{\omega_2}(E) &= \int_E \omega_2 \wedge \omega_2 = \int_{\rho^{-1}(E)} \rho^*(\omega_2 \wedge \omega_2) = \int_B \omega_1 \wedge \omega_1. \end{aligned}$$

We claim that there is no symplectomorphism $\psi : (E, \omega_2) \rightarrow (E, \omega_1)$. For if such ψ existed we could consider the map $\phi = \psi \circ \rho : B \rightarrow E$ which has the property $\phi^* \omega_1 = (\rho^* \circ \psi^*) \omega_1 = \rho^* \omega_2 = \omega_1$.

On the other hand E is contained in the cylinder

$$Z_1\left(\frac{1}{a}\right) = \{x \in \mathbf{R}^4 : x_1^2 + y_1^2 < \frac{1}{a^2}\} \subseteq \mathbf{C}^2$$

and we have obtained an embedding

$$\phi : B \hookrightarrow Z_1\left(\frac{1}{a}\right), \quad \phi^* \omega_1 = \omega_1.$$

But as $a > 1$ by property 3 this is a contradiction to Gromov's theorem.

The deficiency of this construction is that the 2-form $\omega_2 = \rho_*^{-1}\omega_1$ might have discontinuous coefficients at $0 \in E$ as Lempert mappings ρ are generally not smooth at the origin. But the situation can be rescued by pushing the singularity near the boundary with a suitable symplectic mapping.

To be more precise, denote by $B(r)$ the open ball of radius r : $B(r) = \{|z| < r\} \subseteq \mathbf{C}^n$. As $\rho : B \rightarrow E$ is a homeomorphism and $\rho(0) = 0$ we can choose a small neighbourhood V of the origin in E such that $\rho^{-1}(V) \subseteq B(\epsilon)$ for a given ϵ . Applying standard convolution arguments we can smoothen u_2 to a C^∞ plurisubharmonic function \tilde{u}_2 coinciding with u_2 outside of V .

By Stokes' theorem the volume of E with respect to $\tilde{\omega}_2 = dd^c\tilde{u}_2$ will still be the same. Now choose a compactly supported smooth Hamiltonian function H on $B = B(1)$ with

$$H = \begin{cases} x_1 & \text{on } B(1 - 2\epsilon), \\ 0 & \text{on } B \setminus B(1 - \epsilon). \end{cases}$$

Let h be the time- $(1 - 3\epsilon)$ -map of the Hamiltonian vectorfield X_H defined by

$$\iota_{X_H}\omega_1 = dH.$$

As $X_H = \frac{\partial}{\partial y_1}$ on $B(1 - 2\epsilon)$ the symplectomorphism h maps the ball $B(\epsilon)$ around the origin into $B \setminus B(1 - 4\epsilon)$.

Let us now assume the existence of a diffeomorphism $\psi : E \rightarrow E$ with $\psi^*\omega_1 = \tilde{\omega}_2$. As before we put $\phi = \psi \circ \rho : B(1) \rightarrow E$ and now the mapping $\phi \circ h^{-1}|_{B(1-4\epsilon)}$ again yields a symplectic embedding

$$(B(1 - 4\epsilon), \omega_1) \hookrightarrow (Z_1(\frac{1}{a}), \omega_1).$$

Choosing ϵ small enough we arrive again at a contradiction to Gromov's theorem.

Thus we obtain

Theorem 1. *There exist two smooth defining functions u_1 and u_2 of a strictly pseudoconvex domain E such that their 2-forms ω_1, ω_2 give rise to the same volume but they are not symplectically equivalent.* \square

Let us finish this note with the following simple observation:

Proposition 1. *Let $D \subseteq \mathbf{C}^n$ be a complete circular domain with defining function u . If $dd^c u = \omega_1$ then D is a ball.*

Proof. Since $\omega_1 = \sum dy_i \wedge dx_i = dd^c u_0$ for $u_0 = \frac{1}{4} \sum |z_i|^2$ the function $h = u_0 - u$ is pluriharmonic in D and satisfies the following Dirichlet-type problem:

$$(2) \quad \begin{cases} dd^c h = 0, \\ h|_{\partial D} = u_0|_{\partial D}. \end{cases}$$

Let L_p be a complex line through the origin and a point $p \in \partial D$. Then $L_p \cap D = p \cdot \Delta = \{p\zeta : \zeta \in \Delta\}$ as D is complete circular. But $h|_{p \cdot \Delta}$ is harmonic and $h|_{\partial(p \cdot \Delta)} = \frac{1}{4}|p|^2$. Hence $h \equiv \frac{1}{4}|p|^2$, in particular $h(0) = u(0) = \frac{1}{4}|p|^2$. This proves that $|p|^2$ is independent of the choice of p . \square

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