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S^2-BUNDLES OVER ASPHERICAL SURFACES AND 4-DIMENSIONAL GEOMETRIES

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Abstract. Melvin has shown that closed 4-manifolds that arise as S^2-bundles over closed, connected aspherical surfaces are classified up to diffeomorphism by the Stiefel-Whitney classes of the associated bundles. We show that each such 4-manifold admits one of the geometries $S^2 \times E^2$ or $S^2 \times \mathbb{H}^2$ [depending on whether $\chi(M) = 0$ or $\chi(M) < 0$]. Conversely a geometric closed, connected 4-manifold M of type $S^2 \times E^2$ or $S^2 \times \mathbb{H}^2$ is the total space of an S^2-bundle over a closed, connected aspherical surface precisely when its fundamental group $\Pi_1(M)$ is torsion free. Furthermore the total spaces of $\mathbb{RP}^2$-bundles over closed, connected aspherical surfaces are all geometric. Conversely a geometric closed, connected 4-manifold M' is the total space of an $\mathbb{RP}^2$-bundle if and only if $\Pi_1(M') \cong \mathbb{Z}/2\mathbb{Z} \times K$ where K is torsion free.

Introduction

Melvin [Me] has shown that closed 4-manifolds that arise as S^2-bundles over closed aspherical surfaces are classified up to diffeomorphism by the Stiefel-Whitney classes of the associated bundles. Moreover one may construct ξ from $w(\xi)$. See [Me, Structure Lemma]. In particular there are two orientable 4-manifolds that arise as total spaces of S^2-bundles over each closed surface and are distinguished by whether or not $w_2(\xi) = 0$. Ue [Ue] has shown that the total spaces M of such orientable bundles over the surfaces of Euler characteristic $\chi = 0$ have the structures of Seifert 4-manifolds and are geometric.

There are just two S^2-bundles over S^2, the product bundle and $S^2 \tilde{\times} S^2$; the latter is not geometric. There are four S^2-bundles over $\mathbb{RP}^2$, each of which admits an $S^2 \times S^2$ geometry and is distinguishable by its Stiefel-Whitney classes [H1]. There are four S^2-bundles over each closed orientable surface B with Euler characteristic $\chi(B) \leq 0$, six S^2-bundles over the Klein bottle Kb and eight S^2-bundles over each closed non-orientable surface B with $\chi(B) \leq 0$, [Me]. The aim of this paper is to prove the following:

Main Theorem. (1) *A closed $S^2 \times E^2$- or $S^2 \times \mathbb{H}^2$-manifold M is the total space of an S^2-bundle over some aspherical surface precisely when its fundamental group $\Pi_1(M)$ is torsion free. Conversely all total spaces of such bundles are geometric.*

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(2) *A closed $S^2 \times E^2$- or $S^2 \times \mathbb{H}^2$ -manifold M' is the total space of an $\mathbb{RP}^2$-bundle over some aspherical surface precisely when its fundamental group splits as $\Pi_1(M') = \mathbb{Z}/2\mathbb{Z} \times K$, where K is torsion free. Conversely all total spaces of such bundles are geometric.*

Generalities

Bundles ξ will be denoted by $p\colon M \to B$ and will have fibre the 2-sphere S^2 throughout unless otherwise stated. Let E^2 denote the 2-dimensional Euclidean geometry with underlying space $\mathbb{R}^2$. Similarly let $\mathbb{H}^2$ denote the geometry of the hyperbolic plane and $\mathbb{X}^2$ denote either of the above two geometries with underlying space X^2, in arguments that encompass both. Henceforth a manifold referred to as being closed is a compact, connected manifold without boundary.

From the long exact homotopy sequence for a fibration we have that $\Pi_1(M) \cong \Pi_1(B)$; in particular $H_1(M) \cong H_1(B)$ with any coefficients. Also, as p is a fibration, the Euler characteristic $\chi(M) = \chi(S^2)\chi(B) = 2\chi(B)$, so by Poincaré duality with coefficients in $\mathbb{Z}/2\mathbb{Z}$ we obtain $\beta_2(M) = 2$ and therefore $H_2(M;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. In fact $H_2(M;\mathbb{Z}/2\mathbb{Z})$ is generated by the images of a cross-section $\sigma_*(B)$ and the inclusion of the fibre $i_*(S^2)$; as the fibre has zero self intersection and the intersection number of a cross-section and a fibre is 1, they are linearly independent. We shall explicitly construct such cross-sections where needed.

Let ξ be an S^2-bundle. The homomorphism $p^*\colon H^*(B;\mathbb{Z}/2\mathbb{Z}) \to H^*(M;\mathbb{Z}/2\mathbb{Z})$, induced by p, is injective and the Whitney sum formula gives:

(a) $w_1(M) = p^*(w_1(B) + w_1(\xi))$, and

(b) $w_2(M) = p^*(w_2(B) + w_1(B) \smile w_1(\xi) + w_2(\xi))$.

See Melvin [Me], Lemma 1.

The Wu class $v_2(M)$ is characterised by:

$$v_2(M) \smile x = x \smile x, \quad \forall x \in H^2(M;\mathbb{Z}/2\mathbb{Z}).$$

Let $\mathcal{D}\colon H^2(M;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} H_2(M;\mathbb{Z}/2\mathbb{Z})$ be the Poincaré duality isomorphism. Then we may dualize the above equation in terms of intersections of surfaces by:

$$\mathcal{D}(x) \cdot \mathcal{D}(v_2(M)) \equiv \mathcal{D}(x) \cdot \mathcal{D}(x) \bmod 2, \quad \forall \mathcal{D}(x) \in H_2(M),$$

so that $\mathcal{D}(v_2(M))$ is the characteristic element for the intersection form. Together with the Wu relation $v_2(M) = w_2(M) + w_1(M)^2$ we have that $w_2(M) = w_1(M)^2$ precisely when all self intersections in M are even. Moreover

$$\begin{aligned} v_2(M) &= p^*(w_2(B) + w_1(B) \smile w_1(\xi) + w_2(\xi)) + p^*(w_1(B) + w_1(\xi))^2 \\ &= p^*(w_2(B) + w_1(B) \smile w_1(\xi) + w_2(\xi) + w_1(B)^2 + w_1(\xi)^2) \\ &= p^*(w_2(\xi)) \text{ from the Wu relations for } B, \end{aligned}$$

and as p^* is injective we obtain $w_2(\xi) = 0 \Longleftrightarrow v_2(M) = 0$.

If ξ is a geometric S^2-bundle with fundamental group $\Pi < O(3) \times \mathit{Isom}(\mathbb{X}^2)$, then $w_1(\xi)$ is detected by the determinant:

$$\det(g_S) = (-1)^{w_1(\xi)(g_S, g_X)} \quad \text{for all } (g_S, g_X) \in \Pi.$$

Geometric 4-manifolds that arise as total spaces of S^2-bundles over closed surfaces

Before proving that all S^2-bundles over closed aspherical surfaces are geometric we shall determine which geometric 4-manifolds are total spaces of S^2-bundles over closed aspherical surfaces.

Lemma. *Let M be a closed $S^2 \times \mathbb{X}^2$-manifold with fundamental group $\Pi_1(M) < \mathrm{Isom}(S^2 \times \mathbb{X}^2)$ and $\Pi_{\mathbb{X}^2}$ denote the image of $\Pi_1(M)$ in $\mathrm{Isom}(\mathbb{X}^2)$.*

1. *If $\Pi_1(M)$ is torsion free, then it acts freely on $\mathbb{X}^2$ and $M = (S^2 \times \mathbb{X}^2)/\Pi_1(M)$ is the total space of an S^2-bundle over $\mathbb{X}^2/\Pi_{\mathbb{X}^2}$.*
2. *$\Pi_1(M)$ has a torsion free subgroup of index two.*

Proof. (1) Let $g \in \Pi_1(M)$ fix $e \in \mathbb{X}^2$. Then $\forall s \in S^2$ the orbit of $(s, e) \in S^2 \times \mathbb{X}^2$ under $\langle g \rangle$ is homeomorphic to the orbit of $s \in S^2$ by $\langle g \rangle$; moreover the stabilizer of the orbit is trivial as $\Pi_1(M)$ acts fixed point free. Now as S^2 is compact these orbits must be finite so that $|\langle g \rangle| < \infty$ and $\Pi_1(M)$ has torsion. Similarly, the kernel of the projection of $\Pi_1(M)$ onto $\Pi_{\mathbb{X}^2}$ is a finite group which acts freely on S^2.

In particular if $\Pi_1(M)$ is torsion free, then we have the following commutative diagram:

$$\begin{array}{ccc} S^2 \times \mathbb{X}^2 & \overset{\rho}{\longrightarrow\!\!\!\!\rightarrow} & \mathbb{X}^2 \\ \Big\downarrow f & & \Big\downarrow \overline{f} \\ M = (S^2 \times \mathbb{X}^2)/\Pi_1(M) & \overset{p}{- - \rightarrow} & \mathbb{X}^2/\Pi_{\mathbb{X}^2} \end{array}$$

where ρ is the canonical projection and f and $\overline{f}$ are universal covering projections. We show that the fibres of p are S^2. Let $e' = \overline{f}(e) \in \mathbb{X}^2/\Pi_{\mathbb{X}^2}$. Then $p^{-1}(e') = f\rho^{-1}\overline{f}^{-1}(e') = f\rho^{-1}(Orb_{\Pi_{\mathbb{X}^2}}(e)) = f(S^2 \times Orb_{\Pi_{\mathbb{X}^2}}(e))$. Since $\Pi_{\mathbb{X}^2}$ acts freely on $\mathbb{X}^2$, the map $S^2 \times \{e\} \mapsto f(S^2 \times Orb_{\Pi_{\mathbb{X}^2}}(e))$ is a bijection so that $p^{-1}(e') \simeq S^2$. Moreover the projection $\Pi_1(M) \longrightarrow\!\!\!\!\rightarrow \Pi_{\mathbb{X}^2}$ is an isomorphism.

(2) Now suppose that $g = (g_S, g_X)$ is a nontrivial element of $\Pi_1(M)$ of finite order. As g_X is a self homeomorphism of the plane of finite order it fixes a point. Therefore g_S must act freely, and so $g_S = -I$ since $\{I, -I\}$ is the only nontrivial subgroup of $O(3)$ that acts freely and properly discontinuously on S^2. Hence if we denote the projection of $\Pi_1(M)$ onto its geometric factor in $O(3)$ by α, the composite map:

$$Isom(S^2 \times \mathbb{X}^2) > \Pi_1(M) \xrightarrow{\alpha} O(3) \xrightarrow{\mathrm{Det}} \{1, -1\} = \mathbb{Z}/2\mathbb{Z}$$

will have kernel $K = \{I\} \times \Pi_{\mathbb{X}^2} \cong \Pi_{\mathbb{X}^2} < \Pi_1(M)$ of index two and must be torsion free. $\square$

Since the fundamental groups of total spaces of S^2-bundles over closed aspherical surfaces are torsion free, we have proved the following:

Theorem 1. *A closed $S^2 \times E^2$- or $S^2 \times \mathbb{H}^2$-manifold M is the total space of an S^2-bundle over a closed aspherical surface if and only if $\Pi_1(M)$ is torsion free.*

Geometric 4-manifolds that arise as total spaces of $\mathbb{RP}^2$-bundles over closed surfaces

Let M be either a closed $S^2 \times E^2$- or $S^2 \times \mathbb{H}^2$-manifold. Suppose $\Pi_1(M) \cong \mathbb{Z}/2\mathbb{Z} \times K$. By the preceding lemma we know K is torsion free and acts freely and properly discontinuously on X^2, with compact quotient.

Let $g \in \Pi_1(M)$ generate the torsion subgroup. Viewing $\Pi_1(M)$ as a subgroup of $O(3) \times \mathit{Isom}(\mathbb{X}^2)$, we may write $g = (g_S, g_X)$ where $g_S \in O(3)$ and $g_X \in \mathit{Isom}(\mathbb{X}^2)$. Since $g_X^2 = \mathrm{id}_{\mathbb{X}^2}$ the fixed point set $F = \mathit{Fix}(g_X) = \{x \in X^2 | g_X(x) = x\}$ is nonempty, and is either a point, a line, or the whole of X^2. Since g_X commutes with the action of K on X^2, we have $KF = F$, and so K acts freely and properly discontinuously on F. But K is neither trivial nor infinite cyclic, and so we must have $F = X^2$. Hence $g_X = \mathrm{id}_{\mathbb{X}^2}$. It now follows as in part (2) of the above lemma that $g_S = -I$ and so $\Pi_1(M) = \langle -I \rangle \times K < O(3) \times \mathit{Isom}(\mathbb{X}^2)$, the product decomposition corresponding to the geometric factors. Furthermore we have the following commutative diagram:

$$\begin{array}{ccc} S^2 \times \mathbb{X} & \xrightarrow{\ \rho\ } & \mathbb{X} \\ \downarrow f & & \downarrow \bar{f} \\ M & \overset{p}{\dashrightarrow} & \mathbb{X}^2/K \end{array}$$

and M is the total space of an $\mathbb{RP}^2$-bundle over $\mathbb{X}^2/K$, by an argument such as part (1) of the lemma above.

Conversely if M is the total space of an $\mathbb{RP}^2$-bundle over B, then from the long exact homotopy sequence for a fibration we have:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\ i\ } \Pi_1(M) \xrightarrow{\ p_\star\ } \Pi_1(B) \longrightarrow 1 \quad .$$

Moreover $w_1(M)$ restricts nontrivially to the fibre as shown below:

$$\begin{array}{ccc} \Pi_1(M) & \xrightarrow{\ w_1(M)\ } & \mathbb{Z}/2\mathbb{Z} \\ \uparrow i & \diagup\!\!\diagup & \\ \Pi_1(\mathbb{RP}^2) & & \end{array}$$

so that the above sequence splits and we have proved the following:

Theorem 2. *A closed $S^2 \times E^2$- or $S^2 \times \mathbb{H}^2$-manifold M is the total space of an $\mathbb{RP}^2$-bundle over a closed aspherical surface if and only if $\Pi_1(M) \cong \mathbb{Z}/2\mathbb{Z} \times K$, where K is torsion free.*

Now for every total space M of an $\mathbb{RP}^2$-bundle over B, $w_1(M) \neq 0$; therefore M is nonorientable and hence has a unique orientation double cover $\widetilde{M}$. Moreover this orientation cover is the total space of an S^2-bundle over B; the $\mathbb{RP}^2$-bundle being obtained from the S^2-bundle by factoring out the fibre by the antipodal map. That is, we have a bijective correspondence:

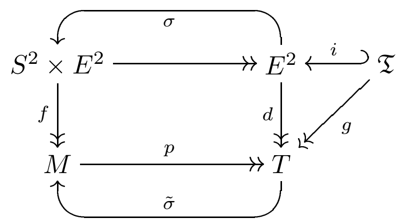
$$\left\{ \begin{array}{c} \text{orientable } S^2\text{-bundles} \\ \text{over } B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathbb{RP}^2\text{-bundles} \\ \text{over } B \end{array} \right\} \quad ,$$

and there are two $\mathbb{RP}^2$-bundles over each closed aspherical surface.

S^2-bundles over T

In the spirit of Ue we show that the nonorientable S^2-bundles over the torus $T = S^1 \times S^1$ and the Klein bottle Kb admit geometries of type $S^2 \times E^2$ before considering the case where $\chi(B) < 0$.

Now we may take the fundamental domain of $T = \mathbb{R}^2/\mathbb{Z}^2$ to be $\mathfrak{T} = \{(s,t) \in \mathbb{R}^2 | 0 \leq s,t \leq 1\}$, and use this to calculate $w_2(\xi)$ by constructing a section $\sigma\colon \mathfrak{T} \to S^2 \times E^2$ of $\mathfrak{T}$ so that σ induces a section $\tilde{\sigma}\colon T \to M$ of ξ as illustrated below:



That is, $\tilde{\sigma}(g(t)) = f\sigma(i(t))\ \ \forall t \in \mathfrak{T}$ where f and d are universal covering projections, and g is the standard gluing map. In fact the cross-sections we shall describe are maps $\sigma\colon \mathfrak{T} \to S^2$, with the action of the gluing map g predetermined by construction.

Let $R_i \in O(3)$ be the reflection in $\mathbb{R}^3$ that changes the sign of the i^{th} coordinate, for $i = 1,2,3$. Then from [H2] we have that the four 4-manifolds that are total spaces of S^2-bundles over T are given by the quotient of $S^2 \times E^2$ by the subgroups of $Isom(S^2 \times E^2) = Isom(S^2) \times Isom(E^2) = O(3) \times (\mathbb{R}^2 \rtimes O(2))$ generated by $(A_j, (1,0), I)$ and $(B_j, (0,1), I)$ for $j = 1,2$, each of which acts freely and cocompactly on $S^2 \times E^2$ and is isomorphic to $\mathbb{Z}^2 = \Pi_1(T) = \Pi_1(M)$, where:

(I) $\xi_1 : A_1 = I, B_1 = -I$;
(II) $\xi_2 : A_2 = R_1, B_2 = R_1 R_2$;
(III) $\xi_3 : A_3 = I, B_3 = I$; and
(IV) $\xi_4 : A_4 = R_1 R_2, B_4 = R_1 R_3$.

Cases (III) and (IV) yield orientable 4-manifolds that are total spaces of S^2-bundles over T and have been shown to be distinguished by $w_2(\xi_3) = 0$ and $w_2(\xi_4) \neq 0$, [Ue].

Using the standard stereographic projection of $S^2 \subset \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ onto $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, we may identify the reflections $R_i\colon S^2 \to S^2$ with the antiholomorphic involutions:

$$z \overset{R_1}{\longmapsto} \frac{1}{\overline{z}}, \quad z \overset{R_2}{\longmapsto} \overline{z}, \quad z \overset{R_3}{\longmapsto} -\overline{z}.$$

Then

$$\sigma_\lambda(s,t) = (2t-1)e^{i\pi(2s+\lambda)} + \lambda t(t-1)$$

defines a 1-parameter family of cross-sections on $\mathfrak{T}$ for ξ_1 with $\sigma_0(s,t) = \sigma_1(s,t)$ only when $(s,t) \in \{(0, \frac{5-\sqrt{17}}{2}), (\frac{1}{2}, \frac{-3+\sqrt{17}}{2})\}$. The tangent plane to $\sigma_0(s,t)$ at $(0, \frac{5-\sqrt{17}}{2})$ is spanned by the 4-vectors $\{(0, 2\pi(4-\sqrt{17}), 1, 0), (2,0,0,1)\}$, and the tangent plane to $\sigma_1(s,t)$ at $(0, \frac{5-\sqrt{17}}{2})$ is spanned by the 4-vectors $\{(0, 2\pi(\sqrt{17}-4), 1, 0), (2-\sqrt{17}, 0, 0, 1)\}$. These two sets together constitute a linearly independent set and hence the intersection at this point is transverse. Similarly the intersection at $(\frac{1}{2}, \frac{-3+\sqrt{17}}{2})$ may be shown to be transverse so that the mod 2 intersection number is $\sigma \cdot \sigma = 0$ [as $\sigma_0(s,t)$ and $\sigma_1(s,t)$ are clearly isotopic]. Hence the second Wu class $v_2(M) = 0$ and from above we get $w_2(\xi_1) = 0$.

Similarly $\sigma_\lambda(s,t) = (\lambda s(s-1)+1)^{(t^2+t-1)} e^{i\pi(2t+\lambda)(2t-1)}$ defines a 1-parameter family of cross-sections on $\mathfrak{T}$ for ξ_2 such that the sections σ_0 and σ_1 intersect transversely in a single point. Hence the intersection number $\sigma \cdot \sigma = 1$, thus $v_2(M) \neq 0$ and $w_2(\xi_2) \neq 0$.

Thus these two bundles are distinguished by their second Stiefel-Whitney classes so that all S^2-bundles over T admit geometries of type $S^2 \times E^2$.

S^2-bundles over Kb

The following six pairs $((A_j,(1,0),\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)),(B_j,(0,1),I))$ for $1 \leq j \leq 6$ of elements of $Isom(S^2 \times E^2)$ each generate a subgroup isomorphic to $\mathbb{Z}\tilde{\times}\mathbb{Z} = \Pi_1(Kb) = \Pi_1(M)$ which acts freely and cocompactly on $S^2 \times E^2$, where:

(I) $\xi_1 : A_1 = I, B_1 = I$;
(II) $\xi_2 : A_2 = I, B_2 = -I$;
(III) $\xi_3 : A_3 = I, B_3 = R_2R_3$;
(IV) $\xi_4 : A_4 = R_1R_2, B_4 = R_1$;
(V) $\xi_5 : A_5 = -I, B_5 = I$; and
(VI) $\xi_6 : A_6 = R_1, B_6 = R_2R_3$.

Remark. The generators given on page 30 of [H1] in the cases (viii) and (ix) are incorrect. The pair (A,B) should be (I, R_2R_3) or (R_1, R_2R_3) respectively.

Cases (V) and (VI) yield orientable 4-manifolds that are total spaces of S^2-bundles over Kb and have been shown to be distinguished by $w_2(\xi_5) = 0$ and $w_2(\xi_6) \neq 0$, [Ue].

Note that in cases (I)-(IV) the generator $(A_j,(1,0),\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right))$ is an orientation reversing homeomorphism of $S^2 \times E^2$ so that the corresponding 4-manifolds are nonorientable. These 4-manifolds are all total spaces of S^2-bundles over Kb, and we shall show that they may be distinguished by the Stiefel-Whitney classes of these bundles.

Since ξ_1 is the product bundle, $w_2(\xi_1) = 0$ and $w_1(\xi_1) = 0$.

Let the fundamental domain of $Kb = \mathbb{R}^2/(\mathbb{Z}\tilde{\times}\mathbb{Z})$ be $\mathfrak{K} = \{(s,t) \in \mathbb{R}^2 | 0 \leq s \leq 1, -\frac{1}{2} \leq t \leq \frac{1}{2}\}$. The functions $\sigma_\lambda(s,t) = (t^2+\frac{3}{4})e^{i\pi(2s+\lambda)} + \lambda(t^2-\frac{1}{4})$ define a 1-parameter family of cross-sections on $\mathfrak{K}$ for ξ_2 that induces a cross-section $\tilde{\sigma}\colon Kb \to M$ of Kb as above with the sections σ_0 and σ_1 disjoint, so that $\sigma \cdot \sigma = 0$. That is, $v_2(M) = 0$ and therefore $w_2(\xi_2) = 0$. Also the orientation character $w_1(\xi_2)$ is nonzero as $\det(-I) = -1$.

The functions $\sigma_\lambda(s,t) = \lambda t e^{i\pi s} + (1-\lambda)(t^2-\frac{1}{4})e^{i\pi 2s}$ define a 1-parameter family of cross-sections for ξ_3 such that the sections σ_0 and σ_1 intersect transversely in one point, so that $\sigma \cdot \sigma = 1$. Hence $v_2(M) \neq 0$ and therefore $w_2(\xi_3) \neq 0$. Furthermore $w_1(\xi_3) = 0$ as $\det(I) = \det(R_2R_3) = 1$.

For ξ_4 the functions $\sigma_\lambda(s,t) = (\lambda(t^2-\frac{1}{4})+1)^{(s^2+s-1)}e^{i\pi(2s+\lambda)(2s-1)}$ define a 1-parameter family of cross-sections with the sections σ_0 and $\sigma_1(s,t)$ intersecting transversely in a point, so that $\sigma \cdot \sigma = 1$. Hence $v_2(M) \neq 0$ and therefore $w_2(\xi_4) \neq 0$. Also $w_1(\xi_4) \neq 0$ as $\det(R_1) = -1$.

Thus all S^2-bundles over Kb are geometric of type $S^2 \times E^2$.

S^2-bundles over closed orientable surfaces with $\chi < 0$

Let T^g be the closed orientable surface of genus g. Then

$$\Pi_1(T^g) = \langle X_1, Y_1, \ldots, X_g, Y_g | \prod_{i=1}^{g} [X_i, Y_i] = 1 \rangle$$

and we have a degree 1 map $\Omega\colon \Pi_1(T^g) \twoheadrightarrow \mathbb{Z}^2$ that kills the generators X_j, Y_j for $j > 1$. If $\mathfrak{T}^g \subset \mathbb{H}^2$ is a $2g$-gon representing the fundamental domain of T^g, then Ω may be induced by a 'collapsing function' $\mathit{\Omega}\colon \mathfrak{T}^g \twoheadrightarrow \mathfrak{T}$ that collapses $2g-4$ sides of $\mathfrak{T}^g$ to a single vertex in the rectangle $\mathfrak{T}$. In turn this induces a degree 1 collapsing function $\widehat{\mathit{\Omega}}\colon T^g \twoheadrightarrow T$ that collapses $g-1$ handles on T^g to a single point on T. Hence given an S^2-bundle ξ, $p\colon M_\alpha \to T$ over T where $M_\alpha = (S^2 \times E^2)/\Gamma_\alpha$ and $\Gamma_\alpha = \{(\alpha(h), h) | h \in \Pi_1(T)\} \leq \mathit{Isom}(S^2 \times E^2)$ where $\alpha\colon \mathbb{Z}^2 \to O(3)$ is as before [there are four choices for α]; we may pull back along $\widehat{\mathit{\Omega}}$ to get the commutative diagram:

$$\begin{array}{ccc} M_{\alpha\Omega} & \longrightarrow & M_\alpha \\ \Big\downarrow{\scriptstyle p_{\alpha\Omega}} & & \Big\downarrow{\scriptstyle p} \\ T^g & \xrightarrow{\widehat{\mathit{\Omega}}} & T \end{array}$$

and an S^2-bundle $\widehat{\mathit{\Omega}}^*(\xi)$, $p_{\alpha\Omega}\colon M_{\alpha\Omega} \to T^g$ over T^g, where $M_{\alpha\Omega} = (S^2 \times \mathbb{H}^2)/\Gamma_{\alpha\Omega}$ and $\Gamma_{\alpha\Omega} = \{(\alpha\Omega(h), h) | h \in \Pi_1(T^g)\} \leq \mathit{Isom}(S^2 \times \mathbb{H}^2)$. As $\widehat{\mathit{\Omega}}$ is of degree 1, it induces monomorphisms in cohomology so that all the Stiefel-Whitney classes are carried from ξ to $\widehat{\mathit{\Omega}}^*(\xi)$, that is, $w(\widehat{\mathit{\Omega}}^*(\xi)) = \widehat{\mathit{\Omega}}^*(\xi) = w(\xi)$. Hence all S^2-bundles over T^g for $g \geq 2$ are geometric of type $S^2 \times \mathbb{H}^2$.

S^2-bundles over closed nonorientable surfaces with $\chi < 0$

Let B be the closed surface $\#^3\mathbb{RP}^2 = T\#\mathbb{RP}^2 = Kb\#\mathbb{RP}^2$. Then we have a collapsing function map $\widehat{\mathit{\Omega}}\colon T\#\mathbb{RP}^2 \twoheadrightarrow \mathbb{RP}^2$ that collapses the torus summand to a single point. This map $\widehat{\mathit{\Omega}}$ is of degree 1 and so induces monomorphisms in cohomology. In particular $\widehat{\mathit{\Omega}}^*$ preserves the orientation character, that is, $w_1(\widehat{\mathit{\Omega}}^*(\xi)) = \widehat{\mathit{\Omega}}^* w_1(\mathbb{RP}^2) = w_1(B)$, and is an isomorphism on H^2. We may pull back the four S^2-bundles ξ over $\mathbb{RP}^2$ along $\widehat{\mathit{\Omega}}$ to obtain the four bundles over B with first Stiefel-Whitney class $w_1(\widehat{\mathit{\Omega}}^*\xi)$ either 0 or $w_1(B)$.

Similarly we have a collapsing function $\widehat{\mathit{\Upsilon}}\colon Kb\#\mathbb{RP}^2 \twoheadrightarrow \mathbb{RP}^2$ that collapses the Klein bottle summand to a single point. This map $\widehat{\mathit{\Upsilon}}$ is of degree 1 mod 2 so that $\widehat{\mathit{\Upsilon}}^* w_1(\mathbb{RP}^2)$ has nonzero square since $w_1(\mathbb{RP}^2)^2 \neq 0$. Note that in this case $\widehat{\mathit{\Upsilon}}^* w_1(\mathbb{RP}^2) \neq w_1(B)$. Hence we may pull back the two S^2-bundles ξ over $\mathbb{RP}^2$ with $w_1(\xi) = w_1(\mathbb{RP}^2)$ to obtain a further two bundles over B with $w_1(\widehat{\mathit{\Upsilon}}^*(\xi))^2 = \widehat{\mathit{\Upsilon}}^* w_1(\xi)^2 \neq 0$, as $\widehat{\mathit{\Upsilon}}$ is a ring monomorphism.

Again there is a collapsing function $\widehat{\mathit{\Delta}}\colon Kb\#\mathbb{RP}^2 \twoheadrightarrow Kb$ that collapses the Klein bottle summand to a single point. Once again $\widehat{\mathit{\Delta}}$ is of degree 1 mod 2 so that we may pull back the two S^2-bundles ξ over Kb with $w_1(\xi) = w_1(Kb)$ along $\widehat{\mathit{\Delta}}$ to obtain the remaining two S^2-bundles over B. These two bundles $\widehat{\mathit{\Delta}}^*(\xi)$ have $w_1(\widehat{\mathit{\Delta}}^*(\xi)) \neq 0$ but $w_1(\widehat{\mathit{\Delta}}^*(\xi))^2 = 0$, as $w_1(Kb) \neq 0$ but $w_1(Kb)^2 = 0$ and $\widehat{\mathit{\Delta}}^*$ is a monomorphism.

Similar arguments apply to bundles over $\#^n\mathbb{RP}^2$ where $n > 3$.

Thus all S^2-bundles over all closed aspherical surfaces are geometric. Furthermore since the antipodal involution of a geometric S^2-bundle is induced by an isometry $(-I, \mathrm{id}_{\mathbb{X}^2}) \in O(3) \times Isom(\mathbb{X}^2)$, we have that all $\mathbb{RP}^2$-bundles over closed aspherical surfaces are geometric. These final two results together with Theorems 1 and 2 prove the main theorem.

An alternative route to the main theorem would be to first show that orientable 4-manifolds which are total spaces of S^2-bundles are geometric, then deduce that $\mathbb{RP}^2$-bundles are geometric [as above], and finally observe that every S^2-bundle space double covers an $\mathbb{RP}^2$-bundle space.

References

[H1] Jonathan A. Hillman, *On 4-manifolds with universal covering space $S^2 \times \mathbb{R}^2$ or $S^3 \times \mathbb{R}$*, Top. Appl. **52** (1993), 23-42. MR **95b:**57020

[H2] Jonathan A. Hillman, *On 4-manifolds with universal covering space a compact geometric manifold*, J. Austral. Math. Soc. (Series A). **55** (1993), 137-148. MR **94i:**57031

[Hu] Dale Husemoller, *Fibre bundles*, Springer-Verlag, New York, 1994. MR **94k:**55001

[Me] Paul Melvin, *2-sphere bundles over compact surfaces*, Proc. Amer. Math. Soc. **92** (1984), 567-572. MR **85j:**57039

[Ue] Masaaki Ue, *Geometric 4-manifolds in the sense of Thurston and Seifert 4-manifolds II*, J. Math. Soc. Japan. **43** (1991), 149-183. MR **91m:**57019

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