

A MONOTONEITY PROPERTY OF THE GAMMA FUNCTION

G. D. ANDERSON AND S.-L. QIU

(Communicated by Hal L. Smith)

ABSTRACT. In this paper we obtain a monotoneity property for the gamma function that yields sharp asymptotic estimates for $\Gamma(x)$ as x tends to ∞ , thus proving a conjecture about $\Gamma(x)$.

1. INTRODUCTION

For real and positive values of x the Euler gamma function Γ and its logarithmic derivative Ψ , the so-called digamma function, are defined as

$$(1.1) \quad \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

For extensions of these functions to complex variables and for basic properties see [WW].

Over the past half century many authors have obtained inequalities for these important functions (see [A1], [A2] and bibliographies in those papers). In keeping with this tradition we here obtain a monotoneity property of the gamma function that yields a sharp asymptotic estimate for $\Gamma(x)$ as x tends to ∞ .

In [AVV, Lemma 2.39] the following result was obtained.

1.2. Lemma.

$$(1.3) \quad \lim_{x \rightarrow \infty} \frac{\log \Gamma(1 + \frac{x}{2})}{x \log x} = \frac{1}{2},$$

and

$$(1.4) \quad f(x) \equiv \frac{1}{x} \log \Gamma(1 + \frac{x}{2})$$

is strictly increasing from $[2, \infty]$ onto $[0, \infty)$.

It was conjectured in [AVV, Remark 2.41] that the function in (1.3) is strictly increasing from $[2, \infty)$ onto $[0, 1/2)$. In order to obtain an affirmative answer to the above conjecture we prove here the following result.

1.5. Theorem. *The function $f(x) \equiv (\log \Gamma(x+1))/(x \log x)$ is strictly increasing from $(1, \infty)$ onto $(1-\gamma, 1)$, where γ is the Euler-Mascheroni constant. In particular, for $x \in (1, \infty)$,*

$$(1.6) \quad x^{(1-\gamma)x-1} < \Gamma(x) < x^{x-1}$$

Received by the editors June 20, 1996.

1991 *Mathematics Subject Classification.* Primary 33B15; Secondary 26A48, 26D07.

Key words and phrases. Gamma function, psi function, monotoneity, inequalities.

and

$$(1.7) \quad \lim_{x \rightarrow \infty} \frac{\log \Gamma(x)}{(x-1) \log(x-1)} = 1.$$

In [AVV] the the following result was also obtained.

1.8. Lemma. *Let $\Omega_n = \pi^{n/2}/\Gamma(1+n/2)$ denote the n -dimensional volume of the unit ball B^n in \mathbb{R}^n . Then*

$$(1.9) \quad \lim_{n \rightarrow \infty} \Omega_n^{1/(n \log n)} = e^{-\frac{1}{2}},$$

$$(1.10) \quad \Omega_n^{1/n} \text{ decreases strictly to } 0 \text{ as } n \rightarrow \infty,$$

$$(1.11) \quad \sum_{n=2}^{\infty} \Omega_n^{1/\log n} \text{ is convergent.}$$

It was pointed out in [AVV, Remark 2.41] that if the function in (1.3) above has the conjectured property this would imply that $\Omega_n^{1/(n \log n)}$ is strictly decreasing for $n \geq 2$. Thus our Theorem 1.5 implies this monotoneity of $\Omega_n^{1/(n \log n)}$ (see Corollary 3.1). It should be observed that Ω_n itself is not monotone [BH, pp. 263, 264] (cf. [SV]).

In this paper we let \mathbb{N} denote the set of positive integers and, for the real number x , let $[x]$ denote the integer satisfying $x - 1 < [x] \leq x$.

2. PRELIMINARY RESULTS

Before establishing the main theorem we need to prove some technical lemmas.

2.1. Lemma. *The function $f(x) \equiv \sum_{n=1}^{\infty} \frac{n-x}{(n+x)^3}$ is positive for $x \in [1, 4)$.*

Proof. Let $u(t, x) = (t-x)/(t+x)^3$, for $t, x \in [1, \infty)$. Then u is strictly decreasing in t on $[2x, \infty)$ for any $x \in [1, \infty)$, since $\partial u/\partial t = 2(2x-t)/(x+t)^4$.

For $k \in \mathbb{N}$ and $x \geq 1$, let $v_k = u(k, x)$. Then $v_k(x)$ is strictly decreasing in k for $k \in \mathbb{N} \cap [2x, \infty)$ and

$$v_k(x) = u(k, x) > u(t, x) \quad \text{for } t > k \geq 2x.$$

Hence, for $k \geq 2x \geq 2$,

$$v_k(x) = \frac{k-x}{(k+x)^3} = \int_k^{k+1} v_k(x) dt > \int_k^{k+1} u(t, x) dt,$$

so that

$$(2.2) \quad \begin{cases} \sum_{n=2[x]+2}^{\infty} v_n(x) > \sum_{n=2[x]+2}^{\infty} \int_n^{n+1} u(t, x) dt = \int_{2[x]+x}^{\infty} \frac{t-x}{(t+x)^3} dt \\ \qquad \qquad \qquad = \int_{2[x]+2}^{\infty} \left[\frac{1}{(t+x)^2} - \frac{2x}{(t+x)^3} \right] dt = \frac{2(1+[x])}{(2[x]+x+2)^2}. \end{cases}$$

It follows from (2.2) that

$$\begin{aligned}
 f(x) &= \sum_{n=1}^{2[x]+1} v_n(x) + \sum_{n=2[x]+2}^{\infty} v_n(x) > \sum_{n=1}^{2[x]+1} v_n(x) + \frac{2(1+[x])}{(2[x]+x+2)^2} \\
 &= \left[\frac{1-x}{(1+x)^3} + \frac{2-x}{(2+x)^3} + \cdots + \frac{[x]-x}{([x]+x)^3} \right] \\
 &\quad + \left[\frac{[x]+1-x}{([x]+x+1)^3} + \frac{[x]+2-x}{([x]+x+2)^3} + \cdots + \frac{2[x]+1-x}{(2[x]+x+1)^3} \right] \\
 &\quad \quad + \frac{2(1+[x])}{(2[x]+x+2)^2} \\
 &\geq \frac{1}{(x+1)^3} [(1+x) + (2-x) + \cdots + ([x]-x)] \\
 &\quad + \frac{1}{(2[x]+x+1)^3} [([x]+1-x) + ([x]+2-x) + \cdots + (2[x]+1-x)] \\
 &\quad \quad + 2 \frac{[x]+1}{(2[x]+x+2)^2} \\
 &= \frac{[x]([x]+1-2x)}{2(x+1)^3} + \frac{([x]+1)(3[x]+2-2x)}{2(2[x]+x+1)^3} + \frac{2[x]+1}{(2[x]+x+2)^2} \\
 &\equiv f_1(x).
 \end{aligned}$$

If $1 \leq x < 2$, then $[x] = 1$ and

$$(2.3) \quad \begin{cases} f_1(x) = \frac{1-x}{(x+1)^3} + \frac{5-2x}{(3+x)^3} + \frac{4}{(4+x)^2} \\ = \frac{1}{(x+1)^3(4+x)^2} [3x^3 + 5x^2 + 4x + 20] + \frac{5-2x}{(3+x)^3} > 0. \end{cases}$$

If $2 \leq x < 3$, then $[x] = 2$ and

$$(2.4) \quad \begin{cases} f_1(x) = \frac{3-2x}{(x+1)^3} + 3 \frac{4-x}{(5+x)^3} + \frac{6}{(6+x)^2} \\ = \frac{4x^3 - 3x^2 - 18x + 114}{(x+1)^3(6+x)^2} + 3 \frac{4-x}{(5+x)^3} \\ > \frac{4x^3 + 33}{(x+1)^3(6+x)^2} + 3 \frac{4-x}{(5+x)^3} > 0. \end{cases}$$

If $3 \leq x < 4$, then $[x] = 3$ and

$$(2.5) \quad \left\{ \begin{aligned} f_1(x) &= \frac{3(2-x)}{(x+1)^3} + \frac{2(11-2x)}{(7+x)^3} + \frac{8}{(8+x)^2} \\ &= \frac{5x^3 - 18x^2 - 72x + 392}{(x+1)^3(8+x)^2} + 2\frac{11-2x}{(7+x)^3} \\ &> \frac{1}{(8+x)^2} \left[\frac{5x^3 - 18x^2 - 72x + 392}{(x+1)^3} + \frac{22-4x}{7+x} \right] \\ &= \frac{x^4 + 27x^3 - 144x^2 - 50x + 2766}{(x+8)^2(x+7)(x+1)^3} \\ &> \frac{1072}{(x+8)^2(x+7)(x+1)^3} > 0. \end{aligned} \right.$$

The conclusion now follows from (2.2) – (2.5). \square

2.6. Lemma. *The function $g(x) \equiv x^2\Psi'(1+x) - x\Psi(1+x) + \log\Gamma(1+x)$ is positive for all $x \in [1, \infty)$.*

Proof. From the well-known difference equation $\Gamma(x+1) = x\Gamma(x)$ [WW, p. 237] it follows easily that

$$(2.7) \quad \Psi(x+1) = \frac{1}{x} + \Psi(x),$$

from which we obtain

$$(2.8) \quad xg'(x) = x^2\Psi'(x) + x^3\Psi''(x) + 1 \equiv g_1(x).$$

Since

$$\Psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2}, \quad \Psi''(x) = -2 \sum_{n=0}^{\infty} \frac{1}{(x+n)^3}$$

[Ah, p. 200, (31)], g_1 can be rewritten as

$$(2.9) \quad g_1(x) = x^2 \sum_{n=1}^{\infty} \frac{n-x}{(n+x)^3} = x^2 f(x),$$

where f is as in Lemma 2.1.

From Lemma 2.1, (2.8), and (2.9), we see that g is strictly increasing on $[1, 4]$, and hence, by [W, Exercise 2, p. 80] and [Ah, p. 199, (29)],

$$(2.10) \quad g(x) \geq g(1) = \Psi'(2) - \Psi(2) = \sum_{n=2}^{\infty} \frac{1}{n^2} - 1 + \gamma = \frac{\pi^2}{6} + \gamma - 2 = 0.2221 \dots > 0$$

for $x \in [1, 4]$.

Next, since

$$(2.11) \quad \left\{ \begin{aligned} \frac{1}{x} &< \Psi'(x) < \frac{1}{x-1}, \\ \log x - \frac{1}{x} &< \Psi(x) < \log x - \frac{1}{2x}, \end{aligned} \right.$$

for $x > 1$ (see [ABRVV, Theorem 3.1], [S, Lemma 4b], [A2, (2.2)]), it follows from (2.7) that

$$(2.12) \quad \begin{cases} g(x) = x^2\Psi'(x) - x\Psi(x) + \log x + \log \Gamma(x) - 2 \\ > x + (1 - x)\log x + \log \Gamma(x) - \frac{3}{2} \equiv g_2(x). \end{cases}$$

Differentiation gives

$$g_2'(x) = \Psi(x) - (\log x - \frac{1}{x}), \quad x > 1,$$

which is positive by (2.11). Hence g_2 is strictly increasing on $[1, \infty)$ so that, for $x \in [4, \infty)$,

$$(2.13) \quad g(x) > g_2(x) \geq g_2(4) = \frac{5}{2} - 3\log 4 + \log 6 = 0.1328 \dots > 0.$$

The result now follows from (2.10) and (2.13). □

2.14. Lemma. *The function $h(x) \equiv x\Psi(1+x) - \log \Gamma(1+x)$ is strictly increasing from $[0, \infty)$ onto $[0, \infty)$. Moreover,*

$$(2.15) \quad \lim_{x \rightarrow \infty} \frac{h(x)}{x} = 1 \quad \text{and} \quad \frac{h(x)}{x} = 1 + O\left(\frac{\log x}{x}\right)$$

as $x \rightarrow \infty$.

Proof. Differentiation gives

$$h'(x) = x\Psi'(x+1) > 0,$$

and the monotonicity of h follows. Clearly, $h(0) = 0$. Since

$$(2.16) \quad \begin{cases} \log \Gamma(x) = (x - \frac{1}{2})\log x - x + \frac{1}{2}\log(2\pi) + O\left(\frac{1}{x}\right), \\ \Psi(x) = \log x - \frac{1}{2x} + O\left(\frac{1}{x^2}\right) \end{cases}$$

as $x \rightarrow \infty$ by [S, Theorems 4, 5], we have

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{h(x)}{x} &= \lim_{x \rightarrow \infty} \left[\log(1+x) - \frac{1}{2(x+1)} - \frac{1}{x}\left(x + \frac{1}{2}\right)\log(x+1) + \frac{x+1}{x} \right] \\ &= \lim_{x \rightarrow \infty} \left[1 - \frac{\log(1+x)}{2x} \right] = 1. \end{aligned}$$

□

2.17. Lemma. *The function $H(x) \equiv \log x - \frac{1}{h(x)}\log \Gamma(x+1)$ is strictly increasing from $[1, \infty)$ onto $[0, 1)$. Here h is as in Lemma 2.14. In particular, for all $x \in (1, \infty)$,*

$$(2.18) \quad 1 - \frac{1}{\log x} < \frac{\log \Gamma(x+1)}{x\Psi(x+1)} < 1 - \frac{1}{1 + \log x}.$$

Proof. Clearly, $H(1) = 0$. It follows from (2.15) and (2.16) that

$$\begin{aligned} \lim_{x \rightarrow \infty} H(x) &= \lim_{x \rightarrow \infty} \left[\left(1 - \frac{x}{h(x)}\right) \log x + \frac{x}{h(x)} \left(-\frac{\log x}{2x} + 1 - \frac{\log(2\pi)}{2x} + O\left(\frac{1}{x^2}\right)\right) \right] \\ &= \lim_{x \rightarrow \infty} \left[O\left(\frac{(\log x)^2}{h(x)}\right) + \frac{x}{h(x)} + O\left(\frac{1}{x}\right) \right] \\ &= 1 + \lim_{x \rightarrow \infty} O\left(\frac{1}{x}(\log x)^2\right) = 1. \end{aligned}$$

Next, by differentiation, we get

$$\begin{aligned} H'(x) &= \frac{1}{x} - \frac{1}{(h(x))^2} [\Psi(x+1)h(x) - x\Psi'(x+1)\log\Gamma(x+1)] \\ &= \frac{1}{x} - \frac{1}{x(h(x))^2} [(x\Psi(x+1) - \log\Gamma(x+1) + \log\Gamma(x+1))h(x) \\ &\quad - x^2\Psi'(x+1)\log\Gamma(x+1)] \\ &= \frac{\log\Gamma(x+1)}{x(h(x))^2} [x^2\Psi'(x+1) - h(x)] \\ &= \frac{g(x)}{x(h(x))^2} \log\Gamma(x+1), \end{aligned}$$

where g is as in Lemma 2.6. Hence the monotonicity of H follows from Lemma 2.6. The inequality (2.18) is clear. \square

3. PROOF OF THE MAIN THEOREM

We now show how Theorem 1.5 follows from the lemmas in Section 2. By differentiation we get

$$(x \log x)^2 f'(x) = h(x)H(x),$$

where $f(x) = (\log\Gamma(x+1))/(x \log x)$ and where h and H are as in Lemmas 2.14 and 2.17, respectively. Hence, the monotonicity of f follows from Lemmas 2.14 and 2.17.

Next, by l'Hôpital's Rule, we have

$$f(1^+) = \lim_{x \rightarrow 1} \frac{\Psi(x+1)}{1 + \log x} = \Psi(2) = -1 - \gamma$$

[AS, 6.3.3] and, by (2.16),

$$\begin{aligned} \lim_{x \rightarrow \infty} f(x) &= \lim_{x \rightarrow \infty} \frac{(x + \frac{1}{2}) \log(x+1) - (x+1) + \frac{1}{2} \log(2\pi)}{x \log x} \\ &= \lim_{x \rightarrow \infty} \frac{\log(x+1)}{\log x} = 1. \end{aligned}$$

Inequality (1.6) and limit (1.7) are clear. \square

3.1. Corollary. (1) *The function $f(x) \equiv (\log \Gamma(1 + \frac{x}{2})) / (x \log x)$ is strictly increasing from $[2, \infty)$ onto $[0, 1/2)$ (the conjecture in [AVV, Remark 2.41] is true).*

(2) *For $n \in \mathbb{N}$, let $\Omega_n = \pi^{n/2} / \Gamma(1 + n/2)$ be the n -dimensional volume of the unit ball B^n in \mathbb{R}^n . Then the sequence $G(n) \equiv \Omega_n^{1/(n \log n)}$ is strictly decreasing for $n \geq 2$, with $G(2) = \pi^{1/\log 4}$ and $\lim_{n \rightarrow \infty} G(n) = e^{-1/2}$.*

Proof. (1) Let $t = x/2$. Then

$$f(x) = \frac{\log t}{2 \log(2t)} \cdot \frac{\log \Gamma(t+1)}{t \log t},$$

and the conclusion follows from Theorem 1.5 since the function $(\log t) / \log(2t)$ is strictly increasing from $[1, \infty)$ onto $[0, 1)$.

(2) Since

$$\log G(n) = \frac{1}{2} \frac{\log \pi}{\log n} - \frac{\log \Gamma(1 + \frac{n}{2})}{n \log n},$$

the assertion follows from part (1). □

3.2. Remark. By methods similar to those used to prove Theorem 1.5 we can show that the function $f(x) \equiv (\log \Gamma(x+1)) / ((x-1) \log(2x))$ is strictly increasing from $[4.5, \infty)$ onto $[c, 1)$, where $c = (\log(135\sqrt{\pi}/32)) / (7 \log 3)$. However, f is not monotone on $(1, \infty)$ since $f'(x) < 0$ when x is near 1.

3.3. Conjecture. $f(x) \equiv (\log \Gamma(x+1)) / (x \log x)$ is concave on $(1, \infty)$.

ACKNOWLEDGEMENTS

This paper was completed during the second author's visit to Michigan State University. The authors thank Dr. M. Vuorinen for helpful comments.

REFERENCES

- [AS] M. Abramowitz and I. A. Stegun, editors, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, 1965. MR **31**:1400
- [Ah] L. V. Ahlfors, *Complex Analysis, An introduction to the theory of analytic functions of one complex variable*, 3rd ed., McGraw-Hill, New York, 1979. MR **80c**:30001
- [A1] H. Alzer, *Some gamma function inequalities*, Math. Comp. **60** (1993), 337-346. MR **93f**:33001
- [A2] H. Alzer, *On some inequalities for the gamma and psi functions*, Math. Comp. **66** (1997), 373-389. MR **97e**:33004
- [ABRVV] G. D. Anderson, R. W. Barnard, K. C. Richards, M. K. Vamanamurthy, and M. Vuorinen, *Inequalities for zero-balanced hypergeometric functions*, Trans. Amer. Math. Soc. **347** (1995), 1713-1723. MR **95m**:33002
- [AVV] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Special functions of quasi-conformal theory*, Exposition. Math. **7** (1989), 97-136. MR **90k**:30032
- [BH] J. Böhm and E. Hertel, *Polyedergeometrie n -dimensionalen Räumen Konstanter Krümmung*, Birkhäuser, Basel - Boston - Stuttgart, 1981. MR **82k**:52001a
- [S] J. Sandor, *Sur la fonction gamma*, Publ. Centre Rech. Math. Pures (I) (Neuchâtel) **21** (1989), 4-7.
- [SV] D. J. Smith and M. K. Vamanamurthy, *How small is a unit ball?*, Math. Mag. **62** (1989), 101-107. MR **90e**:51029

- [W] H. F. Weinberger, *Partial Differential Equations*, John Wiley & Sons, New York, 1965.
[WW] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge Univ. Press, Cambridge, 1958. MR **31**:2375

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN 48824

E-mail address: `anderson@math.msu.edu`

SCHOOL OF SCIENCE AND ARTS, HANGZHOU INSTITUTE OF ELECTRONICS ENGINEERING (HIEE), HANGZHOU 310037, PEOPLE'S REPUBLIC OF CHINA