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THE FULLY INVARIANT SUBGROUPS OF LOCAL WARFIELD GROUPS

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ABSTRACT. We prove that every fully invariant subgroup of a p-local Warfield abelian group is the direct sum of a Warfield group and an S-group. This solves a problem posed some time ago by R. B. Warfield, and finalizes recent work of M. Lane concerning the fully invariant subgroups of balanced projective groups.

1. INTRODUCTION

In this account we will focus on p-local abelian groups, that is, abelian groups G such that $qG = G$ for all rational primes $q \neq p$. Such groups are naturally $\mathbb{Z}_p$-modules, and that will be our point of view in the next section. We write $|x|_G$ for the p-height in G of $x \in G$. In [7], I. Kaplansky calls a group G *fully transitive* if x can be mapped to y by an endomorphism of G whenever $x, y \in G$ satisfy $|p^i y|_G \geq |p^i x|_G$ for all $i \geq 0$. A subgroup of G is called *fully invariant* when it is taken into itself by every endomorphism of G. Fully invariant subgroups of p-local groups are necessarily p-local as well. For reduced, fully transitive p-groups T, Kaplansky proves that any fully invariant subgroup of T has the form $T(\boldsymbol{\sigma}) = \{x \in T : |p^i x|_T \geq \sigma_i \text{ for } i \geq 0\}$ for some sequence $\boldsymbol{\sigma} = (\sigma_0, \sigma_1, \sigma_2, \dots)$ of ordinal numbers and ∞. Because simply presented p-groups are fully transitive, L. Fuchs and E. Walker were able to use Kaplansky's result to prove that fully invariant subgroups of such groups are again simply presented (see [2, p. 101]).

The entire class of simply presented groups and their direct summands is referred to as the class of *Warfield* groups. Since the problem of characterizing the fully invariant subgroups of Warfield groups was posed in [12] and [13], no complete solution has come forth. Perhaps that is due to the fact that Warfield groups need not be fully transitive (see [1]), hence Kaplansky's methods did not carry over to locate fully invariant subgroups within the containing groups. Indeed, recent progress on this problem has been for a special class of Warfield groups which turn out to be fully transitive: the balanced projective groups. We refer the reader to [8] and [9] for further details on this sidelight. Although we lack full transitivity for the groups at hand, we will be able to establish the following result.

Theorem. *Let G be a Warfield group, and H a fully invariant subgroup of G. Then H is the direct sum of a Warfield group and an S-group. If the torsion subgroup*

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of G is simply presented, then H is a Warfield group with simply presented torsion subgroup.

As defined in [12], an S-group is isomorphic to the torsion subgroup of a balanced projective group. Note that the theorem gives an extensive class of Warfield groups that is closed under taking fully invariant subgroups. In case G is torsion, the theorem simply reiterates the result of Fuchs and Walker that was mentioned above.

We remark that both Warfield groups and S-groups can be classified by numerical invariants (see [11]). We will use the facts that torsion subgroups of simply presented groups are S-groups ([5, Theorem 45]), and $T(\boldsymbol{\sigma})$ is an S-group whenever T is one ([9, Corollary 1]). We refer the reader to [3] for basic facts about simply presented, balanced projective and Warfield groups.

2. Full invariance in mixed groups

Let S be a subgroup of a p-group T, and $\ell(T)$ denote the p-length of T. Write $\boldsymbol{\sigma}_{T,S}$ for the ordinal sequence $(\sigma_0, \sigma_1, \sigma_2, \dots)$, where σ_i is the minimum of the set $\{|p^i x|_T : x \in S\} \cup \{\ell(T)\}$. If T is reduced and $f(S) \subseteq S$ for all f in a subring of $\mathrm{End}(T)$ acting fully transitively on T, then the proof of [7, Theorem 25] shows that $S = T(\boldsymbol{\sigma}_{T,S})$. After a lemma, we use this fact to push Kaplansky's result on fully invariant subgroups a little farther.

Lemma 1. *Let $\tau_0 < \cdots < \tau_k$ be ordinals, and G a group. If $x \in G$ satisfies $|x|_G > \tau_k$, there exists $y \in G$ such that $x = p^{k+1}y$, and $|p^i y|_G \geq \tau_i$ for $i \leq k$.*

The proof of the lemma is a routine induction on k. In the following result, tG denotes the torsion subgroup of G.

Proposition 1. *Let G be a reduced, fully transitive group and H a fully invariant subgroup of G. Then $tH = tG(\boldsymbol{\sigma})$ and $H \subseteq G(\boldsymbol{\sigma})$, where $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{tG,tH}$.*

Proof. Let A be the subring of $\mathrm{End}(tG)$ consisting of the endomorphisms of G restricted to tG. Clearly, A acts fully transitively on tG. Since H is fully invariant in G, we have $f(tH) \subseteq tH$ for all $f \in A$. Therefore, as noted above, $tH = tG(\boldsymbol{\sigma})$.

To finish the proof, we must show $H \subseteq G(\boldsymbol{\sigma})$. For the sake of contradiction, suppose $|p^n z|_G < \sigma_n$ for some n and $z \in H$. First assume there exists $m \in \omega$ such that $|p^{n+m+1}z|_G > |p^{n+m}z|_G + 1$. Applying Lemma 1 with $x = p^{n+m+1}z$, $\tau_i = |p^i z|_G + 1$ and $k = n + m$, we obtain $y \in G$ such that $p^{n+m+1}z = p^{n+m+1}y$ and $|p^i y|_G \geq |p^i z|_G + 1$ for $i \leq n + m$. Let $t = z - y \in tG$. Then $|p^i z|_G \leq |p^i t|_G$ for all i, hence $t \in tH$ because G is fully transitive and H is fully invariant in G. But $|p^n t|_G = |p^n z|_G < \sigma_n$, contradicting $tH = tG(\boldsymbol{\sigma})$. Therefore, we may assume that the height sequence of $p^n z$ contains no gaps. Since $|p^n z|_G < \sigma_n \leq \ell(tG)$, we may choose $s \in tG$ of height $|p^n z|_G$. By Lemma 1, there exists $y \in tG$ such that $s = p^n y$ and $|p^i y|_G \geq |p^i z|_G$ for $i \leq n - 1$ (take $y = s$ if $n = 0$). Then $|p^i z|_G \leq |p^i y|_G$ for all i because the height sequence of $p^n z$ is gapless. Hence, $y \in tH$. But $p^n y = s$ has height $|p^n z|_G < \sigma_n$, contradicting $tH = tG(\boldsymbol{\sigma})$. This final contradiction finishes the proof.

It is a fairly immediate consequence of [4, Theorem 3.4] that reduced Warfield groups of rank 1 are fully transitive. (If G is such a group and $x \in G$, then $\langle x \rangle = \mathbb{Z}_p x$ is knice in G and $G/\langle x \rangle$ is a Warfield group; a one-sided version of the theorem in [4] then implies that each homomorphism $\langle x \rangle \longrightarrow G$ that increases heights in G is induced by an endomorphism of G.) Further details about fully transitive mixed groups can be found in [1].

Corollary 1. *Let G be a reduced, simply presented group of rank 1. If H is a fully invariant subgroup of G, H is the direct sum of a Warfield group and an S-group. If tG is simply presented, then H is a Warfield group and tH is simply presented.*

Proof. Since G is simply presented of rank 1 it is fully transitive. Denote $T = tG$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{T,tH}$. By Proposition 1, $tH = T(\boldsymbol{\sigma})$ and $H \subseteq G(\boldsymbol{\sigma})$. Note that $T(\boldsymbol{\sigma})$ is an S-group, and is simply presented if T is. The desired conclusions follow immediately if $H = T(\boldsymbol{\sigma})$ is torsion or $H \cong \mathbb{Z}_p \bigoplus T(\boldsymbol{\sigma})$ is split. For the remaining case, we have $\mathbb{Q} \cong H/T(\boldsymbol{\sigma}) \subseteq G(\boldsymbol{\sigma})/T(\boldsymbol{\sigma})$. This implies $H/T(\boldsymbol{\sigma}) = G(\boldsymbol{\sigma})/T(\boldsymbol{\sigma})$ since $G(\boldsymbol{\sigma})/T(\boldsymbol{\sigma})$ is torsion-free of rank 1. We conclude $H = G(\boldsymbol{\sigma})$. Let $x \in H$ have infinite order. Then $G/\langle x\rangle$ is a simply presented p-group, and it is straightforward to verify (or see [10, Fact F]) that $(G/\langle x\rangle)(\boldsymbol{\sigma}) = (G(\boldsymbol{\sigma}) + \langle x\rangle)/\langle x\rangle = G(\boldsymbol{\sigma})/\langle x\rangle = H/\langle x\rangle$. Hence $H/\langle x\rangle$ is simply presented, and it follows that H is a Warfield group.

We now draw the same conclusions for G of arbitrary rank.

Corollary 2. *Let G be a reduced, simply presented group. If H is fully invariant in G, H is the direct sum of a Warfield group and an S-group. If tG is simply presented, H is a Warfield group and tH is simply presented.*

Proof. We may assume G is nontorsion. There is a decomposition $G = \bigoplus_{i\in I} G_i$ into simply presented groups G_i of rank 1. Because H is fully invariant in G we obtain a corresponding decomposition $H = \bigoplus_{i\in I} H_i$, where each H_i is a fully invariant subgroup of G_i. By Corollary 1, each H_i is the direct sum of a Warfield group and an S-group, and is a Warfield group with simply presented torsion if tG (and hence tG_i) is simply presented. The desired results for H follow immediately.

3. Proof of theorem

Let G be a Warfield group and H a fully invariant subgroup of G. We wish to show that H is the direct sum of a Warfield group and an S-group, and is a Warfield group with simply presented torsion if tG is simply presented. Write $G = G_1 \bigoplus G_2$, where G_1 is reduced and G_2 is divisible. Then $H = H_1 \bigoplus H_2$, with H_i fully invariant in G_i for $i = 1, 2$. It follows that H_2 is the direct sum of bounded groups and divisible groups, hence H_2 and tH_2 are simply presented. We may therefore assume G is reduced. By [5, Theorem 45], there is a decomposition $G = A \bigoplus B$ in which A is balanced projective (hence simply presented) and tB is simply presented. Because H is fully invariant in G we obtain a decomposition $H = A' \bigoplus B'$, where A' and B' are fully invariant subgroups of A and B, respectively. By Corollary 2, A' is the direct sum of a Warfield group and an S-group, and is a Warfield group with simply presented torsion if tG (and hence tA) is simply presented. To finish the proof, we will show that B' is a Warfield group with simply presented torsion. For $i \in \omega$, denote $B'_i = B'$ and $B_i = B$. Let $C' = \bigoplus_{i\in\omega} B'_i$ and $C = \bigoplus_{i\in\omega} B_i$. The inclusions $B'_i = B' \subseteq B = B_i$ induce an inclusion $C' \subseteq C$, under which C' is fully invariant in C. By [6, Corollary 7], C is simply presented. Because tC is simply presented, we conclude from Corollary 2 that C' is a Warfield group with simply presented torsion. Since B' is isomorphic to a direct summand of C', B' is also a Warfield group with simply presented torsion.

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