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A NOTE ON p-HYPONORMAL OPERATORS

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Abstract. Let T be a p-hyponormal operator on a Hilbert space with polar decomposition $T = U|T|$ and let $\widetilde{T} = |T|^t U|T|^{r-t}$ for $r > 0$ and $r \geq t \geq 0$. We study order and spectral properties of $\widetilde{T}$. In particular we refine recent Furuta's result on p-hyponormal operators.

1. Introduction

An operator means a bounded linear transformation from a Hilbert space into itself. For an operator T, let $U|T|$ denote the polar decomposition of T, where U is a partially isometric operator, $|T|$ is a positive square root of T^*T and $N(T) = N(|T|) = N(U)$, where $N(S)$ denotes the kernel of an operator S.

An operator T is said be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ for $1 \geq p > 0$. If $p = 1$, T is called hyponormal, and if $p = \frac{1}{2}$, T is called semi-hyponormal. A p-hyponormal operator $T = U|T|$ is q-hyponormal for $p \geq q$ [11] and $U|T|^p$ is hyponormal. In [15], Xia introduced the class of semi-hyponormal operators and obtained results analogous to those of hyponormal operators. Aluthge [1] studied p-hyponormal operators for $1 \geq p > 0$. In particular, he defined the operator $\widetilde{T} = |T|^{\frac{1}{2}} U|T|^{\frac{1}{2}}$ which is called the Aluthge transformation of T. Aluthge transformations have significant applications (see, e.g., [4], [7], [9], [12]). Recently Furuta [9] extended order properties of Aluthge transformations to those of operators $\widetilde{T} = |T|^q U|T|^q$ with $N(U) = N(U^*)$ (see also Addendum in [9]). In this paper, we refine Furuta's result by dropping this kernel condition. Applying this result, we give a general version of Patel's theorem [12, Theorem 1] on the normality for a p-hyponormal operator. We also study spectral properties of p-hyponormal operators.

Throughout this paper, let $1 \geq p > 0$.

2. Generalized Aluthge transformations

In this section, using Furuta inequality we study order properties of p-hyponormal operators. We also generalize Patel's Theorem [12] on the normality of a p-hyponormal operator.

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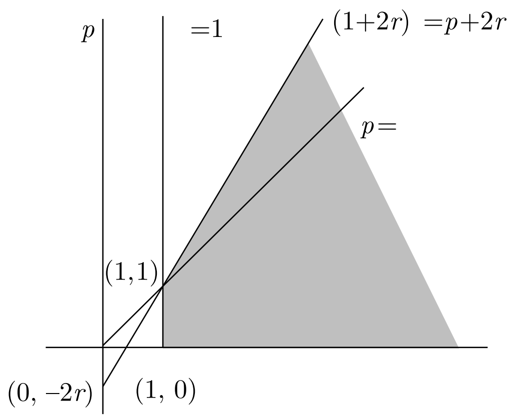
Lemma 1. *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator on a Hilbert space H. Then there exists an isometric operator V satisfying $V|T| = U|T|$ and $|T|V = |T|U$.*

Moreover $\widehat{U} = \begin{pmatrix} V & I - VV^* \\ 0 & -V^* \end{pmatrix}$ *is a unitary operator on $H \oplus H$ such that* $\widehat{U}\begin{pmatrix} |T| & 0 \\ 0 & 0 \end{pmatrix}$ *is p-hyponormal.*

Proof. Proof is based on an idea of [16, p. 41, Lemma 3.5]. Since $N(U) = N(|T|) = N(T) \subseteq N(T^*)$, we define V by $V\xi = U\xi$ for $\xi \in H \ominus N(U)$ and $V\xi = \xi$ for $\xi \in N(U)$. It is easy to see that V has the desired properties.

Furuta established the following result as an extension of Löwner-Heinz inequality.

The Furuta inequality ([8, Theorem 1]). *If $A \geq B \geq 0$, then for each $r \geq 0$,*
(i) $(B^r A^p B^r)^{\frac{1}{q}} \geq (B^r B^p B^r)^{\frac{1}{q}}$
and
(ii) $(A^r A^p A^r)^{\frac{1}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}}$
hold for $p \geq 0$ and $q \geq 1$ with $(1 + 2r)q \geq p + 2r$.



Figure

The domain surrounded by p, q and r in the figure is the best possible one for the Furuta inequality in [14].

Theorem 2. *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator. For $r > 0$ and $r \geq t \geq 0$, let $q = \min\{\frac{p+t}{r}, \frac{p+(r-t)}{r}, 1\}$ and $\widetilde{T} = |T|^t U |T|^{r-t}$. Then $\widetilde{T}$ satisfies that $(\widetilde{T}^*\widetilde{T})^q \geq |T|^{2rq} \geq (\widetilde{T}\widetilde{T}^*)^q$. In particular, $\widetilde{T}$ is q-hyponormal.*

Proof. We first prove that if W is a unitary operator such that $T = W|T|$, then $S = |T|^t W |T|^{r-t}$ is q-hyponormal. This part is close to the proof of [1, Theorem 2] or [9, Theorem 1]. Put

$$A = W^*|T|^{2p}W,\ B = |T|^{2p} \text{ and } C = W|T|^{2p}W^*.$$

Then we have that for any $s > 0$,

$$A^s = W^*|T|^{2sp}W \text{ and } C^s = W|T|^{2sp}W^*.$$

Since T is p-hyponormal, we have $(T^*T)^p \geq (TT^*)^p$, or equivalently

$$A \geq B \geq C.$$

Let $q' = \frac{1}{q}$. Then the Furuta inequality (i) gives

$$(S^*S)^q = (|T|^{r-t}W^*|T|^{2t}W|T|^{r-t})^{\frac{1}{q'}}$$

$$= (B^{\frac{r-t}{2p}}A^{\frac{2t}{2p}}B^{\frac{r-t}{2p}})^{\frac{1}{q'}} \geq (B^{\frac{r-t}{2p}}B^{\frac{2t}{2p}}B^{\frac{r-t}{2p}})^{\frac{1}{q'}} = B^{\frac{r}{pq'}},$$

since $(1 + 2\frac{r-t}{2p})q' \geq \frac{p+r-t}{p}\frac{r}{p+r-t} = \frac{r}{p} = \frac{2t}{2p} + 2\frac{r-t}{2p}$ and $q' \geq 1$.

Similarly, the Furuta inequality (ii) gives

$$B^{\frac{r}{pq'}} = (B^{\frac{t}{2p}}B^{\frac{2(r-t)}{2p}}B^{\frac{t}{2p}})^{\frac{1}{q'}} \geq (B^{\frac{t}{2p}}C^{\frac{2(r-t)}{2p}}B^{\frac{t}{2p}})^{\frac{1}{q'}}$$

$$= (|T|^tW|T|^{2(r-t)}W^*|T|^t)^{\frac{1}{q'}} = (SS^*)^q,$$

since $(1 + 2\frac{t}{2p})q' \geq \frac{p+t}{p}\frac{r}{p+t} = \frac{r}{p} = \frac{2(r-t)}{2p} + 2\frac{t}{2p}$ and $q' \geq 1$.

Hence $(S^*S)^q \geq |T|^{2rq} \geq (SS^*)^q$ and S is q-hyponormal.

Suppose that U is not unitary. By Lemma 1, we choose an isometric operator V such that $V|T| = U|T|$ and $|T|V = |T|U$. Put $\widehat{U} = \begin{pmatrix} V & * \\ 0 & * \end{pmatrix}$, $\widehat{T} = \widehat{U}\begin{pmatrix} |T| & 0 \\ 0 & 0 \end{pmatrix}$, $\widehat{S} = |\widehat{T}|^t\widehat{U}|\widehat{T}|^{r-t}$ and $\widehat{E} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Applying the above argument to $\widehat{T}$, we have

$$(|\widehat{T}|^{r-t}\widehat{U}^*|\widehat{T}|^{2t}\widehat{U}|\widehat{T}|^{r-t})^q = (\widehat{S}^*\widehat{S})^q \geq |\widehat{T}|^{2rq} \geq (\widehat{S}\widehat{S}^*)^q = (|\widehat{T}|^t\widehat{U}|\widehat{T}|^{2(r-t)}\widehat{U}^*|\widehat{T}|^t)^q.$$

Then

$$\widehat{|T|}^{2rq} = \begin{pmatrix} |T|^{2rq} & 0 \\ 0 & 0 \end{pmatrix}, \ (\widehat{S}\widehat{S}^*)^q = \begin{pmatrix} (|T|^tU|T|^{2(r-t)}U^*|T|^t)^q & 0 \\ 0 & 0 \end{pmatrix}.$$

By Hansen's inequality [10],

$$\begin{pmatrix} (|T|^{r-t}U^*|T|^{2t}U|T|^{r-t})^q & 0 \\ 0 & 0 \end{pmatrix} = (\widehat{E}(\widehat{S}^*\widehat{S})\widehat{E})^q \geq \widehat{E}(\widehat{S}^*\widehat{S})^q\widehat{E}.$$

Since $\widehat{E}(\widehat{S}^*\widehat{S})^q\widehat{E} \geq \widehat{E}|\widehat{T}|^{2rq}\widehat{E} = |\widehat{T}|^{2rq}$, we obtain the desired inequalities.

Remark. With the notation of Theorem 2, Aluthge [1] and Furuta [9] considered $\widetilde{T}$ in the greatest q for $r = 1$ and $r \geq p$.

Theorem 3. *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator on a Hilbert space H. For $r > 0$ and $r \geq t \geq 0$, put $\widetilde{T} = |T|^tU|T|^{r-t}$. If $\widetilde{T}$ is normal, then T is normal.*

Proof. There exists $q > 0$ such that $\widetilde{T}$ is q-hyponormal by Theorem 2. Moreover we have that

$$\begin{aligned}(|T|^{r-t}U^*|T|^t|T|^tU|T|^{r-t})^q &= (\widetilde{T}^*\widetilde{T})^q \\ &\geq |T|^{2rq} \\ &\geq (\widetilde{T}\widetilde{T}^*)^q = (|T|^tU|T|^{r-t}|T|^{r-t}U^*|T|^t)^q.\end{aligned}$$

Since $\widetilde{T}$ is normal,

$$(|T|^{r-t}U^*|T|^{2t}U|T|^{r-t})^q = |T|^{2rq} = (|T|^tU|T|^{2(r-t)}U^*|T|^t)^q,$$

that is,

$$(*) \qquad |T|^{r-t}U^*|T|^{2t}U|T|^{r-t} = |T|^{2r} = |T|^tU|T|^{2(r-t)}U^*|T|^t.$$

For an operator X on H, it is easy to see that $N(X) = X^*(H)^\perp$, so that $N(X)^\perp = \overline{X^*(H)}$, the closure of $X^*(H)$. Let s be any positive number. Since $N(|T|^s) = N(|T|)$, it holds that

$$\overline{|T|^s(H)} = \overline{|T|(H)}.$$

Let P denote the orthogonal projection having range $\overline{|T|(H)}$. Since

$$N(|T|^s) = N(|T|) = N(T) \subseteq N(T^*) = N(U^*),$$

we have that

$$(**) \qquad |T|^sP = |T|^s, \ U^*P = U^*.$$

(i) If $r - t > 0$, then the second equality of $(*)$ implies that $|T|^{2r-t}|T|^t = |T|^tU|T|^{2(r-t)}U^*|T|^t$. It follows from $(**)$ that

$$|T|^{2r-t} = |T|^{2r-t}P = |T|^tU|T|^{2(r-t)}U^*P = |T|^tU|T|^{2(r-t)}U^*.$$

Taking adjoints, we have that $|T|^{2r-t} = U|T|^{2(r-t)}U^*|T|^t$. Thus

$$|T|^{2(r-t)} = U|T|^{2(r-t)}U^* = |T^*|^{2(r-t)},$$

so that $|T| = |T^*|$. Therefore T is normal.

(ii) If $r-t = 0$, then $(*)$ implies that $U^*|T|^{2r}U = |T|^{2r} = |T|^rUU^*|T|^r$, and hence $|T|^r = |T|^rP = |T|^rUU^*P = |T|^rUU^*$. Then we have that $N(U^*) \subseteq N(|T|) = N(U)$, so that $UU^* = U^*U$. We obtain that

$$|T^*|^{2r} = U|T|^{2r}U^* = UU^*|T|^{2r}UU^* = |T|^{2r}.$$

Therefore T is normal. This completes the proof.

3. Spectra

In this section, we show a property of the point spectrum of a p-hyponormal operator. Applying this property, we give a spectral mapping theorem for the Weyl spectrum of a p-hyponormal operator.

Throughout this section, let $r > 0$ and $r \geq t \geq 0$.

The following two lemmas are known. We include proofs for completeness.

Lemma 4. *Let A and B be operators and suppose that $A = A^*$. Then AB is invertible if and only if BA is invertible.*

Proof. (i) If AB is invertible, there exists an operator X such that $ABX = I$. Then $(BX)^*A = I$, so that A is invertible. Since AB and A are invertible, so is B. Therefore BA is invertible.

(ii) If BA is invertible, a similar argument implies that A and B are invertible. Therefore AB is invertible.

Lemma 5. *If T is an operator such that $T = V|T|$ with partial isometric operator V, then $\sigma(|T|^tV|T|^{r-t}) = \sigma(V|T|^r)$.*

Proof. It is well-known that $\sigma(|T|^t(V|T|^{r-t})) - \{0\} = \sigma((V|T|^{r-t})|T|^t) - \{0\}$ (see, for example, [3, Proposition 5.3 in Chapter I]). By Lemma 4 we have $\sigma(|T|^tV|T|^{r-t}) = \sigma(V|T|^r)$.

The following is a general version of [5, Theorem 3].

Theorem 6. *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator on a Hilbert space H. Then $\sigma(U|T|^r) = \{e^{i\theta}\rho^r : e^{i\theta}\rho \in \sigma(T)\}$.*

Proof. If U is unitary, the equality holds by [5, Theorem 3]. Suppose that U is not unitary. From Lemma 1 there exists a unitary operator W on $H \oplus H$ such that $W\begin{pmatrix}|T| & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}T & 0\\ 0 & 0\end{pmatrix}$ is p-hyponormal. Put $A = \begin{pmatrix}|T| & 0\\ 0 & 0\end{pmatrix}$. Then we have that $\sigma(T) \cup \{0\} = \sigma(WA)$ and $\sigma(WA^r) = \sigma(U|T|^r) \cup \{0\}$. We choose $q > 0$ such that WA and WA^r are q-hyponormal.

Using [5, Theorem 3], we have that $\sigma(WA^r) = \{e^{i\theta}\rho^r : e^{i\theta}\rho \in \sigma(WA)\}$. Then $\sigma(U|T|^r) - \{0\} = \sigma(WA^r) - \{0\} = \{e^{i\theta}\rho^r : e^{i\theta}\rho \in \sigma(T) - \{0\}\}$, $0 \in \sigma(U|T|^r)$ and $0 \in \sigma(U|T|) = \sigma(T)$. Then $\sigma(U|T|^r) = \{e^{i\theta}\rho^r : e^{i\theta}\rho \in \sigma(T)\}$.

Corollary 7. *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator. Then $\sigma(|T|^tU|T|^{r-t}) = \{e^{i\theta}\rho^r : e^{i\theta}\rho \in \sigma(T)\}$.*

Proof. By Lemma 5 we have that $\sigma(|T|^tU|T|^{r-t}) = \sigma(U|T|^r)$. It follows from Theorem 6 that $\sigma(U|T|^r) = \{e^{i\theta}\rho^r : e^{i\theta}\rho \in \sigma(T)\}$.

Theorem 8. *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator on a Hilbert space H. Then $|T|^tU|T|^{r-t}\xi = e^{i\theta}\rho^r\xi$ if and only if $T\xi = e^{i\theta}\rho\xi$ for $e^{i\theta}\rho \in \mathbf{C}$ and $\xi \in H$.*

Proof. (i) In the case of $\rho = 0$, let $\xi \in N(T)$. Since $\xi \in N(T) = N(U)$, $|T|^tU|T|^{r-t}\xi = 0$. Conversely, if $\xi \in N(|T|^tU|T|^{r-t})$, then $|T|^{r-t}\xi \in N(U)$. Since $T|T|^{r-t}\xi = U|T|^{1+r-t}\xi = 0$ and $1 + r - t > 0$, we have that $\xi \in N(|T|^{1+r-t}) = N(|T|) = N(T)$.

(ii) In the case of $\rho \neq 0$, let $T\xi = e^{i\theta}\rho\xi$. Using [4, Theorem 4] (see also [12, Theorem 2]), we have

$$|T|\xi = \rho\xi \text{ and } U\xi = e^{i\theta}\xi.$$

Then we have $|T|^tU|T|^{r-t}\xi = e^{i\theta}\rho^r\xi$.

Conversely, let $|T|^tU|T|^{r-t}\xi = \beta\xi$, where $e^{i\theta}\rho^r = \beta$. We first assume that $t > 0$. Then there exists a unique vector η in $\overline{T^*(H)}$, the closure of $T^*(H)$, such that

$$|T|^t\eta = \xi.$$

Since $|T|^t\frac{1}{\beta}(U|T|^{r-t}\xi) = \xi$ and $U|T|^{r-t}\xi \in \overline{T(H)} \subseteq \overline{T^*(H)}$, we have

$$\frac{1}{\beta}U|T|^{r-t}\xi = \eta.$$

Then

$$U|T|^r\eta = U|T|^{r-t}|T|^t\eta = U|T|^{r-t}\xi = \beta\eta.$$

It follows from [4, Theorem 4] that

$$|T|^r\eta = |\beta|\eta \text{ and } U\eta = e^{i\theta}\eta.$$

Hence $|T|^r|T|^t\eta = |T|^t|\beta|\eta$, that is, $|T|^r\xi = |\beta|\xi = \rho^r\xi$. Also we have that

$$U\xi = U|T|^t\eta = U|\beta|^{\frac{t}{r}}\eta = |\beta|^{\frac{t}{r}}e^{i\theta}\eta = e^{i\theta}|T|^t\eta = e^{i\theta}\xi.$$

Therefore $T\xi = e^{i\theta}\rho\xi$. If $t = 0$, then $U|T|^r\xi = \beta\xi$. A similar argument implies that $T\xi = e^{i\theta}\rho\xi$.

For an operator T, let $\pi_{00}(T)$ denote the set of isolated eigenvalues of finite multiplicity of T and let $w(T)$ denote the Weyl spectrum, that is,

$$w(T) = \cap\{\sigma(T+K) : K \text{ compact}\}.$$

We need the following two conditions introduced by Baxley [2].

C-1: if $\{\lambda_n\}$ is an infinite sequence of distinct points of the set of eigenvalues of finite multiplicity of T and $\{x_n\}$ is any sequence of corresponding normalized eigenvectors, then the sequence $\{x_n\}$ does not converge.

C-2: if $\lambda \in \pi_{00}(T)$, then $T - \lambda I$ has closed range and index 0, that is,

$$\dim N(T-\lambda I) = \dim(R(T-\lambda I))^\perp < \infty,$$

where $R(T-\lambda I)$ denotes the range of $T - \lambda I$.

Proposition 9. *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator. Then T satisfies C-2.*

Proof. Let $\lambda \in \pi_{00}(T)$. By [4, Theorem 4], $N(T-\lambda I)$ is a reducing subspace for U and $|T|$. Let T_1 and T_2 be the restrictions of T and $U|T|^p$ to $N(T-\lambda I)^\perp$, respectively. Then T_1 is p-hyponormal, $\sigma(T_1) \subseteq \sigma(T)$ and $\sigma(T_2) \subseteq \sigma(U|T|^p)$. By Corollary 7, $\sigma(U|T|)$ and $\sigma(T_1)$ are homeomorphic to $\sigma(U|T|^p)$ and $\sigma(T_2)$, respectively. Suppose that $\lambda \in \sigma(T_1)$. Then $\lambda = e^{i\theta}|\lambda|$ is an isolated point of $\sigma(T_1)$. Since T_2 is hyponormal, it follows from [13, Theorem 2] that $e^{i\theta}|\lambda|^p$ is an eigenvalue of T_2. By Theorem 8, λ is an eigenvalue of T_1. But this is a contradiction since $N(T-\lambda I)^\perp$ contains no eigenvectors of T corresponding to λ. Hence $\lambda \notin \sigma(T_1)$ and we have that

$$R(T-\lambda I) = R(T_1 - \lambda I) = N(T-\lambda I)^\perp.$$

Therefore, $T - \lambda I$ has closed range and index 0.

Chō, Itoh and Ōshiro [6] showed that a p-hyponormal operator T holds Weyl's theorem, that is, $w(T) = \sigma(T) - \pi_{00}(T)$. Chō informed us that Patel gave a different proof by the property $\pi_{00}(T) = \pi_{00}(|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}})$. We give another proof.

Theorem 10 ([6, Theorem]). *If T is a p-hyponormal operator, then $w(T) = \sigma(T) - \pi_{00}(T)$.*

Proof. If $Tx = zx$, then $T^*x = \bar{z}x$ by [4, Theorem 4]. Then T satisfies C-1. By Proposition 9 T satisfies C-2. It follows from [2, Lemmas 3 and 4] that $w(T) = \sigma(T) - \pi_{00}(T)$.

Corollary 11. *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator and put $\widetilde{T} = |T|^tU|T|^{r-t}$. Then $w(\widetilde{T}) = \{e^{i\theta}\rho^r : e^{i\theta}\rho \in w(T)\}$.*

Proof. Using Corollary 7 and Theorem 8, we have that

$$\sigma(\widetilde{T}) = \{e^{i\theta}\rho^r : e^{i\theta}\rho \in \sigma(T)\} \text{ and } \pi_{00}(\widetilde{T}) = \{e^{i\theta}\rho^r : e^{i\theta}\rho \in \pi_{00}(T)\}.$$

It follows from Theorem 2 and Theorem 10 that

$$w(\widetilde{T}) = \sigma(\widetilde{T}) - \pi_{00}(\widetilde{T}).$$

Hence we obtain that $w(\widetilde{T}) = \{e^{i\theta}\rho^r : e^{i\theta}\rho \in w(T)\}$.

We define ψ on $\mathbf{C}$ by $\psi(\rho e^{i\theta}) = \rho^t e^{i\theta}\rho^{r-t}$ and put $\psi(T) = |T|^t U |T|^{r-t}$. Restating Corollary 7, Theorem 8 and Corollary 11, we have the following spectral mapping result:

$$\sigma(\psi(T)) = \psi(\sigma(T)),\ \pi_{00}(\psi(T)) = \psi(\pi_{00}(T)) \text{ and } w(\psi(T)) = \psi(w(T)).$$

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Addendum

(1) We would like to cite the following result in Addendum of [9];

Theorem 1′. *Let $T = U|T|$ be the polar decomposition of p-hyponormal for $1 \geq p > 0$ with $N(T) = N(T^*)$. Then $\widetilde{T} = |T|^s U |T|^t$ is $\frac{p+s}{s+t}$-hyponormal for any $s \geq 0$ and $t \geq \max\{p, s\}$.*

(2) After this paper was written, the author have found Aluthge's paper; "Some generalized theorems on p-hyponormal operators, Integral Equations and Operator Theory, **24** (1996), 497–501", in which he proved a theorem (Theorem 1) closely related to our Theorem 2. In fact, our theorem implies his theorem.

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