

A GENERALIZATION OF THE DE BRANGES THEOREM

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ABSTRACT. In this paper a generalization of de Branges' proof of the Bieberbach conjecture is given. The argument does not make use of the Askey-Gasper theorem.

1. INTRODUCTION

It is well known that in 1984 the American mathematician de Branges [1], [2] proved the famous Bieberbach conjecture, thus, solving completely a problem that had challenged many mathematicians over a period of seventy years.

In this paper, the author gives a generalization of de Branges' proof. The new methods used are quite elementary and yield de Branges' result without using the Askey-Gasper theorem [3].

2. RESULT

In this section we shall give the main theorem and its proof.

Theorem. *Let*

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$$

and

$$(2) \quad \log \frac{f(z)}{z} = \sum_{k=1}^{\infty} C_k z^k, \quad |z| < 1.$$

If for any real number set $\{\lambda_k\}_{k=1}^n$, $\lambda_k \geq 0$ ($k = 1, 2, \dots, n$), the condition

$$\lambda_k + 2 \sum_{p=k+1}^n (-1)^{p-k} \lambda_p \geq 0$$

is satisfied, then

$$(3) \quad \sum_{k=1}^n k \lambda_k |C_k|^2 \leq 4 \sum_{k=1}^n \frac{\lambda_k}{k}, \quad n = 1, 2, \dots$$

Equality holds if and only if $f(z)$ is a rotation of the Koebe function.

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Corollary. *In the theorem, set $\lambda_k = n - k + 1$ ($k = 1, 2, \dots, n$), then it becomes de Branges theorem.*

Instead of the de Branges special function system, we consider the function system $S_k(t)$ ($k = 1, 2, \dots, n$) defined by the following system of differential equations:

$$(4) \quad \begin{cases} \frac{S'_{k+1}(t)}{k+1} + \frac{S'_k(t)}{k} = S_{k+1}(t) - S_k(t), \\ S_{n+1}(t) \equiv 0, \\ S_k(0) = \lambda_k, \lambda_k \geq 0 \\ (k = 1, 2, \dots, n). \end{cases}$$

For a given set λ_k ($k = 1, 2, \dots, n$), (4) and (5) define a unique system of functions $\{S_k(t)\}_{k=1}^n$. Now we find expressions for the $S_k(t)$.

First, we can obtain from (4)

$$(6) \quad \frac{S'_k(t)}{k} = 2 \sum_{m=k+1}^n (-1)^{m-k-1} S_m(t) - S_k(t) \quad (k = 1, 2, \dots, n).$$

The characteristic equation of this system of differential equations is

$$(7) \quad \Delta(\lambda) = \begin{vmatrix} 1 + \lambda & -2 \cdot 1 & 2 \cdot 1 & \dots & (-1)^{n-1} 2 \cdot 1 \\ 0 & 2 + \lambda & -2 \cdot 2 & & (-1)^{n-2} 2 \cdot 2 \\ 0 & 0 & 3 + \lambda & (-1)^{n-3} 2 \cdot 3 & \\ \dots & & \dots & & \\ 0 & 0 & 0 & \dots & n + \lambda \end{vmatrix} = 0.$$

So its characteristic roots are

$$(8) \quad \lambda_1 = -1, \lambda_2 = -2, \dots, \lambda_n = -n.$$

Hence we can suppose

$$(9) \quad S_k(t) = \sum_{l=k}^n C_{k,l} e^{-lt} \quad (k = 1, 2, \dots, n),$$

where $C_{k,l}$ ($l = k, k + 1, \dots, n$) are constants.

Lemma 1. *The functions $S_k(t)$ ($k = 1, 2, \dots, n$) in (9) have forms*

$$(10) \quad S_k(t) = C_{k,k} e^{-kt} + \sum_{p=k+1}^n \alpha_{k,p} C_{p,p} e^{-pt},$$

where $\alpha_{k,p}$, $C_{p,p}$ ($p = k, k + 1, \dots, n$) are constants and

$$(11) \quad \begin{cases} \alpha_{k,p} = \frac{(-1)^{p-k} 2k(2p-1)(2p-2) \cdots (p+k+1)}{(p-k)!} \\ (p = k, k + 1, k + 2, \dots, n), \\ \alpha_{k,k} = 1, \end{cases}$$

$$(12) \quad \sum_{m=k}^p (-1)^{m-k} \alpha_{m,p} = \frac{(-1)^{p-k} (2p-1)(2p-2) \cdots (p+k)}{(p-k)!}.$$

Proof. Substituting (9) in (6), we have

$$S'_k(t) + kS_k(t) = 2k \sum_{m=k+1}^n (-1)^{m-k-1} \left(\sum_{l=m}^n C_{m,l} e^{-lt} \right).$$

The solution of this differential equation is

$$S_k(t) = C_{k,k} e^{-kt} + 2k \sum_{p=k+1}^n \left[\sum_{q=k+1}^p (-1)^{q-k} \frac{C_{q,p}}{p-k} \right] e^{-pt}.$$

Therefore,

(13)

$$C_{k,p} = \frac{2k}{p-k} \sum_{q=k+1}^p (-1)^{q-k} C_{q,p} \quad (k = 1, 2, \dots, n-1; p = k+1, k+2, \dots, n).$$

From (13) it may be obtained that

$$C_{k,p} = \alpha_{k,p} C_{p,p}, C_{k+1,p} = \alpha_{k+1,p} C_{p,p}, \dots, C_{p-1,p} = \alpha_{p-1,p} C_{p,p} \\ (k = 1, 2, \dots, n-1; p = k+1, k+2, \dots, n).$$

When $k = p-1$, (13) is

$$C_{p-1,p} = -2(p-1)C_{p,p},$$

and $C_{p-1,p} - C_{p,p} = -(2p-1)C_{p,p}$. Namely (11) and (12) are valid when $k = p-1$.

Now suppose (11) and (12) are valid if $k = m+1, m+2, \dots, p-1$. Then by (13)

$$C_{m,p} = -\frac{2m}{p-m} (C_{m+1,p} - C_{m+2,p} + \dots + (-1)^{p-m-1} C_{p,p}) \\ = -\frac{2m}{p-m} \cdot \frac{(-1)^{p-m-1} (2p-1)(2p-2) \dots (p+m+1)}{(p-m-1)!} C_{p,p} \\ = \frac{(-1)^{p-m} 2m(2p-1)(2p-2) \dots (p+m+1)}{(p-m)!} C_{p,p}.$$

Hence (11) is true. Also,

$$C_{m,p} - C_{m+1,p} + \dots + (-1)^{p-m} C_{p,p} \\ = \left[\frac{(-1)^{p-m} 2m(2p-1)(2p-2) \dots (p+m+1)}{(p-m)!} \right. \\ \left. - \frac{(-1)^{p-m-1} (2p-1)(2p-2) \dots (p+m+1)}{(p-m-1)!} \right] C_{p,p} \\ = \frac{(-1)^{p-m} (2p-1)(2p-2) \dots (p+m)}{(p-m)!} C_{p,p},$$

and (12) is true as well. This completes the proof of Lemma 1. □

Lemma 2. For any positive integer $n = 1, 2, \dots$, the identity

$$(14) \quad \sum_{m=0}^n (-1)^m \frac{m^k}{m!(n-m)!} = 0$$

holds, where $k = 0, 1, 2, \dots, n-1$.

Proof. By the binomial theorem

$$(1-x)^n = n! \sum_{m=0}^n (-1)^m \frac{x^m}{m!(n-m)!}.$$

Apply the operator $x \frac{d}{dx}$ to the $(1-x)^n$ k times, then let $x = 1$. This gives (14). \square

Lemma 3. In (10) the coefficients $C_{k,k}$ have the forms

$$(15) \quad C_{k,k} = \sum_{p=k}^n (-1)^{p-k} \delta_{k,p} \lambda_p \quad (k = 1, 2, \dots, n),$$

where

$$(16) \quad \delta_{k,p} = \begin{vmatrix} \alpha_{k,k+1} & \alpha_{k,k+2} & \cdots & \alpha_{k,p-1} & \alpha_{k,p} \\ 1 & \alpha_{k+1,k+2} & \cdots & \alpha_{k+1,p-1} & \alpha_{k+1,p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & \alpha_{p-1,p} \end{vmatrix} \\ = \frac{(-1)^{p-k} 2k(2k+1)(2k+2) \cdots (p+k-1)}{(p-k)!} \quad (p > k), \\ \delta_{k,k} \equiv 1.$$

Proof. It follows by (5) and (10) that

$$(17) \quad \begin{cases} C_{1,1} + \alpha_{1,2} C_{2,2} + \alpha_{1,3} C_{3,3} + \cdots + \alpha_{1,n} C_{n,n} = \lambda_1, \\ C_{2,2} + \alpha_{2,3} C_{3,3} + \cdots + \alpha_{2,n} C_{n,n} = \lambda_2, \\ \cdots \quad \quad \quad \cdots \quad \quad \quad \cdots \\ C_{n,n} = \lambda_n. \end{cases}$$

We then obtain from (17)

$$C_{k,k} = \begin{vmatrix} \lambda_k & \alpha_{k,k+1} & \cdots & \alpha_{k,n-1} & \alpha_{k,n} \\ \lambda_{k+1} & 1 & \cdots & \alpha_{k+1,n-1} & \alpha_{k+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_{n-1} & 0 & \cdots & 1 & \alpha_{n-1,n} \\ \lambda_n & 0 & \cdots & 0 & 1 \end{vmatrix} \\ = \sum_{p=k}^n (-1)^{p-k} \delta_{k,p} \lambda_p \quad (k = 1, 2, \dots, n).$$

Calculating gives

$$\delta_{p-1,p} = -\frac{2p-2}{1!}, \\ \delta_{p-2,p-1} = -\frac{2p-4}{1!}, \delta_{p-2,p} = \frac{(2p-4)(2p-3)}{2!}.$$

Suppose

$$\delta_{m,p} = \frac{(-1)^{p-m} 2m(2m+1)(2m+2) \cdots (p+m-1)}{(p-m)!} \\ (m = k+1, k+2, \dots, p-1; m < p \leq n).$$

Then by (16)

$$\begin{aligned}
 \delta_{k,p} &= \sum_{m=k+1}^p (-1)^{m-k-1} \alpha_{k,m} \delta_{m,p} \\
 &= \sum_{m=k+1}^p (-1)^{p-m-1} \frac{2k(2m-1)(2m-2) \cdots (m+k+1)}{(m-k)!} \\
 (18) \quad &\quad \cdot \frac{2m(2m+1)(2m+2) \cdots (p+m-1)}{(p-m)!} \\
 &= \sum_{m=k+1}^p (-1)^{p-m-1} \frac{2k(p+m-1)(p+m-2) \cdots (m+k+1)}{(m-k)!(p-m)!}.
 \end{aligned}$$

Let $m - k = \mu$; then (18) becomes

$$\begin{aligned}
 \delta_{k,p} &= \sum_{\mu=1}^{p-k} (-1)^{p-k-\mu-1} \frac{2k(p+k+\mu-1)(p+k+\mu-2) \cdots (2k+\mu+1)}{\mu!(p-k-\mu)!} \\
 &= (-1)^{p-k} \frac{2k(2k+1)(2k+2) \cdots (p+k-1)}{(p-k)!} \\
 &\quad + \sum_{\mu=0}^{p-k} (-1)^{p-k-\mu-1} \frac{2k(2k+\mu+1)(2k+\mu+2) \cdots (p+k+\mu-1)}{\mu!(p-k-\mu)!}.
 \end{aligned}$$

For fixed p and k , the numerator of second term in the last formula is a $(p-k-1)$ th degree polynomial of μ . It must be equal to zero by Lemma 2. Hence

$$\delta_{k,p} = (-1)^{p-k} \frac{2k(2k+1)(2k+2) \cdots (p+k-1)}{(p-k)!}. \quad \square$$

Lemma 4. *Let the functions $S_k(t)$ ($k = 1, 2, \dots, n$) be defined by the system of differential equations (4), (5). Then*

$$(19) \quad -\frac{1}{k} S'_k(0) = \lambda_k + 2 \sum_{p=k+1}^n (-1)^{p-k} \lambda_p.$$

Proof. It follows by the definition of $S_k(t)$ and (5) that

$$-\frac{1}{n} S'_n(0) = \lambda_n, \quad -\frac{1}{n-1} S'_{n-1}(0) = \lambda_{n-1} - 2\lambda_n.$$

Now suppose

$$(20) \quad -\frac{1}{m} S'_m(0) = \lambda_m + 2 \sum_{p=m+1}^n (-1)^{p-m} \lambda_p$$

when $m = k + 1, k + 2, \dots, n$. However, by (10)

$$\begin{aligned} -\frac{1}{k+1}S'_{k+1}(0) &= C_{k+1,k+1} + \sum_{p=k+2}^n \frac{p}{k+1} \alpha_{k+1,p} C_{p,p} \\ &= C_{k+1,k+1} + \sum_{p=k+2}^n \alpha_{k+1,p} C_{p,p} + \sum_{p=k+2}^n \frac{p-k-1}{k+1} \alpha_{k+1,p} C_{p,p} \\ &= \lambda_{k+1} + \sum_{p=k+2}^n \frac{p-k-1}{k+1} \alpha_{k+1,p} C_{p,p}. \end{aligned}$$

Therefore

$$(21) \quad \sum_{p=k+2}^n \frac{p-k-1}{k+1} \alpha_{k+1,p} C_{p,p} = 2 \sum_{p=k+2}^n (-1)^{p-k-1} \lambda_p.$$

On the other hand, by (10)

$$\begin{aligned} -\frac{1}{k}S'_k(0) &= C_{k,k} + \sum_{p=k+1}^n \frac{p}{k} \alpha_{k,p} C_{p,p} \\ &= C_{k,k} + \sum_{p=k+1}^n \alpha_{k,p} C_{p,p} + \sum_{p=k+1}^n \frac{p-k}{k} \alpha_{k,p} C_{p,p} \\ (22) \quad &= \lambda_k + \sum_{p=k+1}^n \frac{p-k}{k} \alpha_{k,p} C_{p,p} \\ &= \lambda_k - 2C_{k+1,k+1} + \sum_{p=k+2}^n \frac{p-k}{k} \alpha_{k,p} C_{p,p}. \end{aligned}$$

But we have

$$\frac{p-k}{k} \alpha_{k,p} = \frac{(-1)^{p-k} 2(2p-1)!}{(p-k-1)!(p+k)!} = -\frac{p+k+1}{k+1} \alpha_{k+1,p}.$$

So from (21) and (22)

$$\begin{aligned} -\frac{1}{k}S'_k(0) &= \lambda_k - 2C_{k+1,k+1} - \sum_{p=k+2}^n \frac{p+k+1}{k+1} \alpha_{k+1,p} C_{p,p} \\ &= \lambda_k - 2 \left(C_{k+1,k+1} + \sum_{p=k+2}^n \alpha_{k+1,p} C_{p,p} \right) - \sum_{p=k+2}^n \frac{p-k-1}{k+1} \alpha_{k+1,p} C_{p,p} \\ &= \lambda_k - 2\lambda_{k+1} - 2 \sum_{p=k+2}^n (-1)^{p-k-1} \lambda_p \\ &= \lambda_k + 2 \sum_{p=k+1}^n (-1)^{p-k} \lambda_p. \end{aligned}$$

Thus (19) is valid. \square

Proof of the Theorem. We replace de Branges special system of functions by the special system of functions $\{S_k(t)\}_{k=1}^n$ and use his method to complete the proof. \square

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