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CESÀRO TRANSFORMS OF FOURIER COEFFICIENTS OF L^∞-FUNCTIONS

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Abstract. In this note, we show that Cesàro transforms of Fourier cosine or sine coefficients of any $L^\infty(0,\pi)$-function are Fourier cosine or sine coefficients of some $BMO(0,\pi)$-function.

Let $p \in [1,\infty)$ and $L^p(0,\pi)$ denote the space of Lebesgue measurable functions $f:(0,\pi)\to(-\infty,\infty)$ with the usual norm $\|f\|_p<\infty$. As is well known, $L^\infty(0,\pi)$, the space of essentially bounded functions $f:(0,\pi)\to(-\infty,\infty)$ with the usual norm $\|f\|_\infty<\infty$, is viewed as a limit space $L^p(0,\pi)$ as $p\to\infty$ in sense of duality. However, in the situation of Hardy space, $L^\infty(0,\pi)$ is substituted by $BMO(0,\pi)$—the space of functions $f\in L^1(0,\pi)$ with bounded mean oscillation:

$$\|f\|_* = \sup \frac{1}{|I|}\int_I |f(x)-f_I|dx < \infty,$$

where the supremum is taken over all subintervals I of $(0,\pi)$, f_I stands for the mean value of f on I: $1/|I|\int_I f(x)dx$ and $|I|$ denotes the length of I: $|I|=\int_I dx$.

The following inclusion chain is helpful for us to understand the relation between those spaces mentioned above:

$$L^\infty(0,\pi) \subsetneq BMO(0,\pi) \subsetneq \bigcap_{1\le p<\infty} L^p(0,\pi).$$

Now, suppose that $f\in L^1(0,\pi)$ and $a=\{a_n\}$ or $b=\{b_n\}$ is the sequence of Fourier cosine or sine coefficients of f extended to $(-\pi,\pi)$ as an even or odd function, namely,

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx, n=0,1,2,...,$$

or

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx, n=1,2,....$$

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In other words, the even or odd extension of $f \in L^1(0,\pi)$ has a Fourier series below:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

or

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$$

The Cesàro transform of $a = \{a_n\}$ or $b = \{b_n\}$ is defined by $\mathcal{C}a = \{A_n\}$ or $\mathcal{C}b = \{B_n\}$, where

$$A_0 = a_0, A_n = \frac{\sum_{k=1}^{n} a_k}{n}, n = 1, 2, ...,$$

or

$$B_n = \frac{\sum_{k=1}^{n} b_k}{n}, n = 1, 2, ...$$

A very natural question is raised here: If $f \in L^p(0,\pi)$ with Fourier cosine or sine coefficients $a = \{a_n\}$ or $b = \{b_n\}$ then must $\mathcal{C}a = \{A_n\}$ or $\mathcal{C}b = \{B_n\}$ be Fourier cosine of sine coefficients of a function also in $L^p(0,\pi)$?

In 1928, Hardy gave a positive answer for the question in the case: $p \in [1,\infty)$, [3]. Since then, there have been some further generalizations, [1], [2]. But there has been no satisfactory result about the case: $p = \infty$, just like the case $p \in [1,\infty)$, [4]. For instance, if taking a bounded function $f(x) = \cos x$ with Fourier cosine coefficients $a = \{0, 1, 0, 0, ...\}$, then we immediately find that $\mathcal{C}a = \{0, 1, 1/2, 1/3, ...\}$ is the sequence of Fourier cosine coefficients of function $F(x) = \log 1/(2\sin(x/2))$. However, this F is unbounded, i.e., $F \notin L^\infty(0,\pi)$, Through a careful observation, we, on the other hand, discover that the function F is of BMO property, that is to say, $F \in BMO(0,\pi)$. More importantly, we are motivated by the above argument to enable us to answer the question in the case of $p = \infty$.

Theorem. *Let $f \in L^\infty(0,\pi)$ with Fourier cosine or sine coefficients $a = \{a_n\}$ or $b = \{b_n\}$. Then $\mathcal{C}a = \{A_n\}$ or $\mathcal{C}b = \{B_n\}$ are Fourier cosine or sine coefficients of some function $F \in BMO(0,\pi)$.*

Proof. It is sufficient to verify this fact for Fourier cosine coefficients.

First of all, we define a linear operator σ on $L^\infty(0,\pi)$, which may be called the Cesàro operator on $L^\infty(0,\pi)$, and is exactly given by

$$(\sigma f)(x) = \int_x^\pi \frac{f(t)}{\tan\frac{t}{2}} dt, f \in L^\infty(0,\pi).$$

Also let

$$(\lambda f)(x) = 2\int_x^\pi \frac{f(t)}{t} dt, f \in L^\infty(0,\pi).$$

Then $\lambda f - \sigma f$ is a bounded function, i.e., $\lambda f - \sigma f \in L^\infty(0,\pi)$. In fact, for $f \in L^\infty(0,\pi)$, it is easy to get that $|(\lambda f)(x) - (\sigma f)(x)| \leq C_1\|f\|_\infty$, where $C_1 = 2\int_0^\pi |\frac{1}{2\tan\frac{t}{2}} - \frac{1}{t}|dt < \infty$.

Next, assuming that $K(t) = -\log|2\sin(t/2)|$ for $t \in (-\pi,\pi)$, we get that $c = \{c_n\}$, where $c_0 = 0, c_n = a_n/n, n = 1, 2, ...$, is the sequence of Fourier cosine coefficients of function $g(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K(t)dt$ [7, p.180]. As Hardy showed in [3], σa is the sequence of Fourier cosine coefficients of function $F(x) = ((\sigma f)(x) + g(x))/2$.

Finally, we prove that $F \in BMO(0,\pi)$. For this end, it suffices to check that λf is in $BMO(0,\pi)$ since $\lambda f - \sigma f \in L^{\infty}(0,\pi)$ and $\|g\|_{\infty} \le C_2\|f\|_{\infty}$, where $C_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\log|2\sin\frac{t}{2}||dt < \infty$. At this time, taking any interval $I = (\alpha,\beta) \subset (0,\pi)$ and $C_I = (\lambda f)(\beta)$ for $f \in L^{\infty}(0,\pi)$, we obtain that $|I| = \beta - \alpha$ and

$$\begin{aligned}\int_I |(\lambda f)(x) - C_I|dx &= \int_{\alpha}^{\beta} |\int_x^{\beta} \frac{f(t)}{t}dt|dx \\ &\le \|f\|_{\infty}\int_{\alpha}^{\beta} \log\frac{\beta}{x}dx \\ &\le |I|\|f\|_{\infty}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\frac{1}{|I|}\int_I |(\lambda f)(x) - (\lambda f)_I|dx &\le \frac{2}{|I|}\int_I |(\lambda f)(x) - C_I|dx \\ &\le 2\|f\|_{\infty}.\end{aligned}$$

That is to say, $\lambda f \in BMO(0,\pi)$. Hence the proof is completed. □

Remarks. 1. $L^{\infty}(0,\pi)$ in Theorem cannot be replaced by $BMO(0,\pi)$. Otherwise, it will follow that $\log^2|x|$ is a function in $BMO(0,\pi)$, which results in a contradiction. Indeed, if $f \in BMO(0,\pi)$ then the statement that $\mathcal{C}a$ or $\mathcal{C}b$ is a sequence of Fourier cosine or sine coefficients of some function in $BMO(0,\pi)$ holds if and only if the operator λ is bounded from $BMO(0,\pi)$ to $BMO(0,\pi)$. Yet, if picking $f(x) = \log(\pi/x)$ then we see that $(\lambda f)(x) = \log^2(\pi/x)$ is outside $BMO(0,\pi)$ due to the unboundedness of $\log(\pi/x)$ on $(0,\pi)$ and Stegenga's multiplier theorem applied to this (λf), [5]. Of course, we have here used a fact that $F \in BMO(0,\pi)$ once $f \in BMO(0,\pi)$, which is easily derived. Actually, if we write $H_R^1(-\pi,\pi)$ and $BMO(-\pi,\pi)$ as the real Hardy space and BMO (bounded mean oscillation) space on $(-\pi,\pi)$ respectively then Fefferman's duality theorem tells us that $[H_R^1(-\pi,\pi)]^* = BMO(-\pi,\pi)$ under the inner pair: $\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$, [6]. To show that $F \in BMO(0,\pi)$, we only need to prove that the following function

$$F_*(x) = \begin{cases} F(x), x \in (0,\pi), \\ 0, x \in (-\pi,0), \end{cases}$$

is in $BMO(-\pi,\pi)$. For this, by Fubini's theorem we find a constant C_3 such that for any $G \in H_R^1(-\pi,\pi)$,

$$\begin{aligned}|\int_{-\pi}^{\pi} F_*(x)G(x)dx| &= \frac{1}{\pi}|\int_0^{\pi}[\int_{-\pi}^{\pi} f(x+t)G(x)dx]K(t)dt| \\ &\le C_3\|f\|_*\|G\|_1\int_0^{\pi} |K(t)|dt.\end{aligned}$$

Equivalently, $F_* \in BMO(-\pi,\pi)$ and hence $F \in BMO(0,\pi)$.

2. From the above proof it is turns out that σ is bounded linear operator from $L^\infty(0,\pi)$ (not from $BMO(0,\pi)$) to $BMO(0,\pi)$. This operator looks very much like the conjugate operator below:

$$\tilde{f}(x) = \frac{1}{\pi}\int_0^\pi \frac{f(x-t)-f(x+t)}{2\tan\frac{t}{2}}dt.$$

Nevertheless, we should note that $\tilde{f} \in BMO(0,\pi)$ if $f \in BMO(0,\pi)$.

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