PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 125, Number 12, December 1997, Pages 3481–3488
S 0002-9939(97)04045-8

INVARIANTS OF SKEW DERIVATIONS

JEFFREY BERGEN AND PIOTR GRZESZCZUK

(Communicated by Ken Goodearl)

Abstract. If σ is an automorphism and δ is a σ-derivation of a ring R, then the subring of invariants is the set $R^{(\delta)} = \{r \in R \mid \delta(r) = 0\}$. The main result of this paper is

Theorem. *Let δ be a σ-derivation of an algebra R over a commutative ring K such that*

$$\delta^{n+k}(r) + a_{n-1}\delta^{n+k-1}(r) + \cdots + a_1\delta^{k+1}(r) + a_0\delta^k(r) = 0,$$

for all $r \in R$, where $a_{n-1}, \dots, a_1, a_0 \in K$ and ${a_0}^{-1} \in K$.
(i) *If $R^{n+1} \neq 0$, then $R^{(\delta)} \neq 0$.*
(ii) *If L is a δ-stable left ideal of R such that $l.ann_R(L) = 0$, then $L^{(\delta)} \neq 0$.*

This theorem generalizes results on the invariants of automorphisms and derivations.

If R is a ring with an automorphism σ, we say that an additive map $\delta : R \longrightarrow R$ is a σ-derivation if

$$\delta(rs) = \delta(r)s + \sigma(r)\delta(s),$$

for all $r, s \in R$. We define the subring of invariants to be the set

$$R^{(\delta)} = \{r \in R \mid \delta(r) = 0\}.$$

It was shown in [HN] that algebraic automorphisms always act with nonzero invariants on nonnilpotent algebras. The analogous result for algebraic derivations was proven in [B]. The simplest examples of σ-derivations are ordinary derivations, which occur when σ is the identity map, as well as maps of the form $1-\sigma$. Therefore the results in this paper generalize results on the invariants of automorphisms and derivations. However, the results on automorphisms and derivations were obtained using group-graded rings, whereas our arguments are entirely combinatorial. In fact, we will present an example in which the 0-eigenspace of a σ-derivation is not a subring, thus the techniques of group-graded rings cannot be applied to this more general situation. Since we would like to apply our results to prove that various subrings and one-sided ideals contain nonzero invariants, we will not be assuming that our rings have a unit element.

Received by the editors December 29, 1995 and, in revised form, July 2, 1996.
1991 *Mathematics Subject Classification.* Primary 16W20, 16W25, 16W55.
The first author was supported by the University Research Council at DePaul University. Both authors were supported by Polish KBN Grant 2 PO3A 050 08. Much of this work was done when the first author was a visitor at the University of Warsaw, Białystok Division and the second author was a visitor at DePaul University. We would like to thank both universities for their hospitality.

We now define the terms we will be using throughout this paper. R will be an algebra over a commutative ring K and the automorphism σ and σ-derivation δ will be assumed to be K-linear transformations. We will be assuming that δ is algebraic over K. By this we mean that

$$\delta^{n+k}(r) + a_{n-1}\delta^{n+k-1}(r) + \cdots + a_1\delta^{k+1}(r) + a_0\delta^k(r) = 0,$$

for all $r \in R$, where $a_{n-1}, \ldots, a_1, a_0 \in K$ and ${a_0}^{-1} \in K$. We let $t : R \longrightarrow R$ be defined as

$$t = \delta^n + \cdots + a_1\delta + a_0.$$

Let

$$R_0 = \{r \in R \mid \delta^m(r) = 0, \text{ for some } m \geq 1\};$$

since a_0 is invertible in K, a standard argument from linear algebra implies that the restriction $t : R_0 \longrightarrow R_0$ is surjective. Therefore, it is now clear that

$$t^j(R) = t(R) = R_0 = \{r \in R \mid \delta^k(r) = 0\},$$

for all $j \geq 1$. If $k = 1$, then we say that δ is separable and in this special case, t maps R onto $R^{(\delta)}$.

We should point out that we will be neither making any assumptions on whether σ is algebraic, nor assuming that there exists any additional relationship between σ and δ. We will make use of the following notation: if A, B, C are subsets of R then $AB^{\#}C$ is the span over K of the elements from the union of the sets B, AB, BC, and ABC. If A is any subset of R, then we let $l.ann_R(A) = \{r \in R \mid rA = 0\}$. Subsets B of R with the properties that $\sigma(B) = B$ and $\delta(B) \subseteq B$ are known, respectively, as σ-stable and δ-stable. Subsets satisfying both properties are called (σ,δ)-stable. For any $A \subseteq R$, we let $A^{(\delta)} = A \cap R^{(\delta)}$. A ring with no nonzero nilpotent (σ,δ)-stable ideals is called (σ,δ)-semiprime and a ring with no nonzero nilpotent σ-stable ideals is called σ-semiprime. Note that in the special case where σ is algebraic, semiprime and σ-semiprime are equivalent.

Lemma 1. *For any δ-stable left ideal L of R,*

$$\sigma^n(L)\sigma^{n-1}(L)\cdots\sigma(L)L \subseteq Rt(L)^{\#}L.$$

Proof. Since L is δ-stable, $t(L)$ is a δ-stable subspace of L. By the definition of t, it follows that if $x \in L$, then there exist $y \in L$ and $l \in t(L)$ such that

$$x = \delta(y) + l.$$

We will prove, by induction on m, that for any $0 \leq m \leq n$ and for every $x_1, \ldots, x_m \in L$, there exist $b_0, b_1, \ldots, b_{n-m-1} \in R$ such that

$$(*) \qquad \sigma^n(x_1)\sigma^{n-1}(x_2)\cdots\sigma^{n-m+1}(x_m)\delta^{n-m}(x) + \sum_{j=0}^{n-m-1} b_j\delta^j(x) \in Rt(L)^{\#}L,$$

for all $x \in L$.

Note that for every $x \in L$,

$$\delta^n(x) + \sum_{j=0}^{n-1} a_j\delta^j(x) = t(x) \in Rt(L)^{\#}L.$$

Thus the $m = 0$ case is done. Next, assume that $n > m \geq 0$ and, by the induction hypothesis, we may assume that (*) holds. If $x \in L$, let $y \in L$ such that $x = \delta(y)+l$, where $l \in t(L)$. Then replacing x in (*) by $x_{m+1}y$, where $x_{m+1} \in L$, yields

$$\begin{aligned}&\left(\sigma^n(x_1)\sigma^{n-1}(x_2)\cdots\sigma^{n-m+1}(x_m)\delta^{n-m}(x_{m+1}) + \sum_{j=0}^{n-m-1} b_j\delta^j(x_{m+1})\right) y\\ &\quad + \sum_{j=0}^{n-m-2} c_j\delta^{j+1}(y)\\ &\quad + \sigma^n(x_1)\sigma^{n-1}(x_2)\cdots\sigma^{n-m+1}(x_m)\sigma^{n-m}(x_{m+1})\delta^{n-m}(y) \in Rt(L)^{\#}L,\end{aligned}$$

for some $c_j \in R$.

By the induction hypothesis,

$$\left(\sigma^n(x_1)\sigma^{n-1}(x_2)\cdots\sigma^{n-m+1}(x_m)\delta^{n-m}(x_{m+1}) + \sum_{j=0}^{n-m-1} b_j\delta^j(x_{m+1})\right) y \in Rt(L)^{\#}L,$$

therefore

$$\begin{aligned}&\sigma^n(x_1)\sigma^{n-1}(x_2)\cdots\sigma^{n-m+1}(x_m)\sigma^{n-m}(x_{m+1})\delta^{n-m}(y)\\ &\quad + \sum_{j=0}^{n-m-2} c_j\delta^{j+1}(y) \in Rt(L)^{\#}L.\end{aligned} \tag{**}$$

Replacing $\delta(y)$ by $x - l$ in (**), shows that

$$\begin{aligned}&\sigma^n(x_1)\sigma^{n-1}(x_2)\cdots\sigma^{n-m+1}(x_m)\sigma^{n-m}(x_{m+1})\delta^{n-m-1}(x) + \sum_{j=0}^{n-m-2} c_j\delta^j(x)\\ &\quad + \sigma^n(x_1)\sigma^{n-1}(x_2)\cdots\sigma^{n-m+1}(x_m)\sigma^{n-m}(x_{m+1})\delta^{n-m-1}(l)\\ &\quad + \sum_{j=0}^{n-m-2} c_j\delta^j(l) \in Rt(L)^{\#}L.\end{aligned}$$

However, $l \in t(L)$, and thus

$$\begin{aligned}&\sigma^n(x_1)\sigma^{n-1}(x_2)\cdots\sigma^{n-m+1}(x_m)\sigma^{n-m}(x_{m+1})\delta^{n-m-1}(l)\\ &\quad + \sum_{j=0}^{n-m-2} c_j\delta^j(l) \in Rt(L)^{\#}L.\end{aligned}$$

As a result, we now have

$$\begin{aligned}&\sigma^n(x_1)\sigma^{n-1}(x_2)\cdots\sigma^{n-m+1}(x_m)\sigma^{n-m}(x_{m+1})\delta^{n-m-1}(x)\\ &\quad + \sum_{j=0}^{n-m-2} c_j\delta^j(x) \in Rt(L)^{\#}L.\end{aligned}$$

The proof of (*) is now complete and the proof of the lemma follows by letting $m = n$ in (*). □

We can now prove our first main result. It is worth noting that in part (ii) of the following theorem, we prove the existence of nonzero invariants in L even though L is not necessarily σ-stable.

Theorem 2. *Let δ be a σ-derivation of an algebra R over a commutative ring K such that*

$$\delta^{n+k}(r)+a_{n-1}\delta^{n+k-1}(r)+\cdots+a_1\delta^{k+1}(r)+a_0\delta^k(r)=0,$$

for all $r\in R$, where $a_{n-1},\ldots,a_1,a_0\in K$ and ${a_0}^{-1}\in K$.

(i) *If $R^{n+1}\neq 0$, then $R^{(\delta)}\neq 0$.*

(ii) *If L is a δ-stable left ideal of R such that $l.ann_R(L)=0$, then $L^{(\delta)}\neq 0$.*

Proof. For (i), let $L=R$ in Lemma 1. It then follows that if $R^{n+1}\neq 0$, then $t(R)\neq 0$. Since δ acts nilpotently on $t(R)$, it is clear that $R^{(\delta)}\neq 0$. For (ii), if $l.ann_R(L)=0$ then $l.ann_R(\sigma^i(L))=0$, for all i. Therefore $\sigma^n(L)\sigma^{n-1}(L)\cdots\sigma(L)L\neq 0$. By Lemma 1, we have that $t(L)\neq 0$ and since L is δ-stable, it follows that $L^{(\delta)}\neq 0$. ☐

We continue with

Lemma 3. *Let $f(m)=\sum_{i=1}^{m}(n+1)^i$ and let $T=t(R)$. If T is a subring of R and if T^l is δ-stable, for all $l\geq 1$, then*

$$R^{f((m-1)k+1)}\subseteq R(T^m)^{\#}R,$$

for any $m\geq 1$.

Proof. We proceed by induction on m. The $m=1$ case follows by letting $L=R$ in Lemma 1, as we have

$$R^{f(1)}=R^{n+1}\subseteq Rt(R)^{\#}R=RT^{\#}R.$$

Now suppose that $m\geq 1$ such that

$$R^{f((m-1)k+1)}\subseteq R(T^m)^{\#}R.$$

Note that if A is any δ-stable subring of R, then

$$t(RA)\subseteq t(R)A+R\delta(A)=TA+R\delta(A). \tag{***}$$

We now define a collection of δ-stable left ideals as follows: $L_0=RT^m+T^m$ and $L_{j+1}=Rt(RL_j)+t(RL_j)$, for $j\geq 0$. We claim that, for any $j\geq 1$,

$$L_j\subseteq RT^{m+1}+T^{m+1}+R\delta^j(T^m).$$

We proceed by induction on j. For $j=1$, we have

$$t(RL_0)\subseteq t(RT^m)\subseteq T^{m+1}+R\delta(T^m),$$

and so,

$$L_1\subseteq RT^{m+1}+T^{m+1}+R\delta(T^m).$$

Now let $j\geq 1$; since T^m and T^{m+1} are δ-stable, it follows by (***) and the induction hypothesis on j that

$$\begin{aligned}t(RL_j)&\subseteq t(RT^{m+1}+R\delta^j(T^m))\\&\subseteq TT^{m+1}+R\delta(T^{m+1})+T\delta^j(T^m)+R\delta^{j+1}(T^m)\\&\subseteq RT^{m+1}+T^{m+1}+R\delta^{j+1}(T^m).\end{aligned}$$

Thus, we now have

$$L_{j+1} = Rt(RL_j) + t(RL_j) \subseteq RT^{m+1} + T^{m+1} + R\delta^{j+1}(T^m),$$

as desired.

Next, we define a sequence of integers $g(j)$ as

$$g(j) = (n+1)^j f((m-1)k+1) + \sum_{i=1}^{j} (n+1)^i.$$

It then follows that $g(0) = f((m-1)k+1)$ and $g(j+1) = (g(j)+1)(n+1)$, for $j \geq 0$. We claim that

$$R^{g(j)} \subseteq R{L_j}^{\#}R,$$

for $j \geq 0$ and we will proceed by induction on j. The $j = 0$ case follows by the induction hypothesis on m as

$$R^{g(0)} = R^{f((m-1)k+1)} \subseteq R(T^m)^{\#}R \subseteq R{L_0}^{\#}R.$$

Now suppose that $j \geq 0$ such that $R^{g(j)} \subseteq R{L_j}^{\#}R$ holds. By the induction hypothesis on j and the surjectivity of σ, for all $i \geq 0$, we have

$$(****) \qquad R^{g(j)+1} = RR^{g(j)} = R\sigma^i(R^{g(j)}) \subseteq R\sigma^i(R{L_j}^{\#}R) \subseteq R(R\sigma^i(L_j)^{\#}R).$$

It now follows from (****) and Lemma 1 that

$$\begin{aligned}
&(R^{g(j)+1})^{n+1} \\
&\quad \subseteq (\sigma^n(RL_j)R + \sigma^n(RL_j))(\sigma^{n-1}(RL_j)R + \sigma^{n-1}(RL_j)) \cdots (RL_jR + RL_j) \\
&\quad \subseteq \sigma^n(RL_j)\sigma^{n-1}(RL_j) \cdots RL_jR + \sigma^n(RL_j)\sigma^{n-1}(RL_j) \cdots RL_j \\
&\quad \subseteq Rt(RL_j)^{\#}R \subseteq R{L_{j+1}}^{\#}R.
\end{aligned}$$

Since $g(j+1) = (g(j)+1)(n+1)$, this implies that

$$R^{g(j+1)} \subseteq R{L_{j+1}}^{\#}R,$$

as desired.

As a result, for any $j \geq 0$, we have

$$R^{g(j)} \subseteq R{L_j}^{\#}R \subseteq R(T^{m+1})^{\#}R + R(\delta^j(T^m))^{\#}R.$$

Since T is a subring, $\delta^k(T^m) \subseteq \delta^k(T) = 0$. Therefore if we let $j = k$, we have

$$R^{g(k)} \subseteq R(T^{m+1})^{\#}R.$$

However, it is easy to see that $g(k) = f(mk+1)$, thus

$$R^{f(mk+1)} \subseteq R(T^{m+1})^{\#}R,$$

as desired, thereby concluding the proof. □

In Lemma 3, we assumed that $t(R)$ is a subring. We now give an example which shows that $t(R)$ need not be a subring. Since $t(R)$ is the 0-eigenspace of R, this illustrates why it was necessary to prove the results in this paper without using group-graded rings.

Example 4. *A finite-dimensional simple ring R with a σ-derivation δ such that $\delta^4 = \delta^2$ and $\sigma^2 = 1$, but $t(R)$ is not a subring.*

Let S be a finite-dimensional simple ring with a noncentral idempotent e. Let $R = S_2$, the 2×2 matrices over S, and let $A = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Next, let σ be the inner automorphism of R induced by B and let δ be the inner σ-derivation of R induced by A. More precisely, if $\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in R$, then

$$\begin{aligned}\delta\begin{pmatrix} r & s \\ t & u \end{pmatrix} &= \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} et & eu - ue \\ 0 & -se \end{pmatrix}.\end{aligned}$$

If we let d denote the inner derivation of S induced by e, then we note that $d^2(s)e = -d(s)e$, for all $s \in S$. Therefore, it is easy to see that

$$\delta\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} et & d(u) \\ 0 & -se \end{pmatrix}, \ \delta^2\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} 0 & -d(s)e \\ 0 & -d(u)e \end{pmatrix},$$

and

$$\delta^4\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} 0 & d^2(s)e \\ 0 & d^2(u)e \end{pmatrix} = \begin{pmatrix} 0 & -d(s)e \\ 0 & -d(u)e \end{pmatrix}.$$

As a result, it is clear that $\delta^4 = \delta^2$ and $\sigma^2 = 1$. We can observe that

$$t(R) = \{\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in R \mid d(s)e = d(u)e = 0\}.$$

Since S is simple, there exist $s_1, s_2 \in S$ such that $(e-1)s_1es_2e \neq 0$. Furthermore, since $d(es_2)e = 0$, it follows that

$$\begin{pmatrix} s_1 & 0 \\ 0 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & es_2 \\ 0 & 0 \end{pmatrix} \in t(R).$$

However,

$$\begin{pmatrix} s_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & es_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & s_1es_2 \\ 0 & 0 \end{pmatrix}$$

and $d(s_1es_2)e = (e-1)s_1es_2e \neq 0$. Thus

$$\begin{pmatrix} 0 & s_1es_2 \\ 0 & 0 \end{pmatrix} \notin t(R)$$

and so, $t(R)$ is not a subring.

In the next lemma we will show that if $t(R)$ is σ-stable, then $t(R)$ must be a subring. In this case, it easily follows that $t(R)^l$ is δ-stable, for all $l \geq 1$, thus the hypotheses of Lemma 3 are satisfied. One large class of σ-derivations with the property that $t(R)$ is σ-stable is q-skew derivations. A q-skew derivation is a σ-derivation with the property that there exists some invertible $q \in K$ such that $\delta\sigma = q\sigma\delta$.

Lemma 5. *If $t(R)$ is σ-stable, then $t(R)$ is a subring of R.*

Proof. Let $r, s \in t(R)$; then we have

$$\delta^{k^2}(rs) = \sum_{i=0}^{k^2} F_{i,\delta,\sigma}(r)\delta^i(s),$$

where $F_{i,\delta,\sigma}$ is a noncommutative polynomial in δ and σ. Each monomial in $F_{i,\delta,\sigma}$ is of degree k^2 such that δ appears $k^2 - i$ times and σ appears i times. Therefore, whenever $i < k$, every monomial in $F_{i,\delta,\sigma}$ must contain a string in which δ appears at least k consecutive times. Since $t(R)$ is stable under both σ and δ and $\delta^k(t(R)) = 0$, it now follows that $F_{i,\delta,\sigma}(r) = 0$, for all $i < k$. However, since $\delta^i(s) = 0$, for all $i \geq k$, we see that $F_{i,\delta,\sigma}(r)\delta^i(s) = 0$, for all $i \leq k^2$. Thus $\delta^{k^2}(rs) = 0$ and so, $rs \in t(R)$. □

We can now prove our second main result.

Theorem 6. *Let δ be a σ-derivation of an algebra R over a commutative ring K such that $t(R)$ is σ-stable and*

$$\delta^{n+k}(r) + a_{n-1}\delta^{n+k-1}(r) + \cdots + a_1\delta^{k+1}(r) + a_0\delta^k(r) = 0,$$

for all $r \in R$, where $a_{n-1}, \ldots, a_1, a_0 \in K$ and ${a_0}^{-1} \in K$.

(i) *If $t(R)$ is nilpotent, then R is nilpotent. In particular, if $t(R)^m = 0$ then*

$$R^{f((m-1)k+1)} = 0, \text{ where } f(m) = \sum_{i=1}^{m}(n+1)^i.$$

(ii) *If R is (σ, δ)-semiprime, then $t(R)$ is (σ, δ)-semiprime.*

Proof. Since $t(R)$ is σ-stable, it follows by Lemma 5 that $t(R)$ satisfies the hypotheses of Lemma 3. The proof of (i) now follows immediately from Lemma 3. For (ii), if $t(R)$ is not (σ, δ)-semiprime, then $t(R)$ contains a (σ, δ)-stable ideal $I \neq 0$ such that $I^2 = 0$. Therefore RI is a (σ, δ)-stable left ideal of R and so, RI is not nilpotent. Hence, by Lemma 3, $t(RI)$ is also not nilpotent. However, by (***) in the proof of Lemma 3,

$$t(RI) \subseteq t(R)I + R\delta(I) \subseteq I + R\delta(I).$$

Since I is δ-stable, continuing in the manner, we see that

$$t(RI) = t^j(RI) \subseteq I + R\delta^j(I),$$

for all $j \geq 1$. Recall that $\delta^k(I) \subseteq \delta^k(t(R)) = 0$; therefore, by letting $j = k$, it follows that

$$t(RI) = t^k(RI) \subseteq I.$$

Thus

$$(t(RI))^2 \subseteq I^2 = 0,$$

which contradicts the fact that $t(RI)$ is not nilpotent. Thus $t(R)$ is (σ, δ)-semiprime. □

For the special case where δ is separable, we can sharpen Theorem 6 and we record this as

Corollary 7. *Let* δ *be a separable* σ*-derivation of a ring* R*; that is,*

$$\delta^{n+1}(r) + a_{n-1}\delta^n(r) + \cdots + a_1\delta^2(r) + a_0\delta(r) = 0,$$

for all $r \in R$*, where* $a_{n-1}, \ldots, a_1, a_0 \in K$ *and* ${a_0}^{-1} \in K$*.*

(i) *If* $R^{(\delta)}$ *is nilpotent, then* R *is nilpotent. In particular, if* $(R^{(\delta)})^m = 0$ *then*

$$R^{f(m)} = 0, \text{ where } f(m) = \sum_{i=1}^{m} (n+1)^i.$$

(ii) *If* R *is* (σ, δ)*-semiprime and if* $R^{(\delta)}$ *is* σ*-stable, then* $R^{(\delta)}$ *is* σ*-semiprime.*

Proof. Since $R^{(\delta)} = t(R)$ is a subring and $(R^{(\delta)})^l$ is δ-stable, for all $l \geq 1$, the proof of (i) follows directly from Lemma 3 with $k = 1$. We now observe that $R^{(\delta)}$ is clearly (σ, δ)-semiprime if and only if it is σ-semiprime. Therefore, (ii) is merely the special case of Theorem 6(ii) with $k = 1$. □

We conclude this paper with an example which shows that the hypothesis in Corollary 7(ii) that $R^{(\delta)}$ be σ-stable is necessary.

Example 8. *A finite-dimensional simple ring* R *with a* σ*-derivation* δ *such that* $\delta^2 = \delta$ *and* $\sigma^2 = 1$*, but* $R^{(\delta)}$ *is not semiprime.*

Let S be a finite-dimensional simple ring and let $R = S_2$, the 2×2 matrices over S. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and then let σ be the inner automorphism of R induced by B and let δ be the inner σ-derivation of R induced by A. More precisely, if $\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in R$, then

$$\delta \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -t \\ t & u-r \end{pmatrix}.$$

It is easy to see that $\delta^2 = \delta$ and $\sigma^2 = 1$. In addition, we can now observe that

$$R^{(\delta)} = \{ \begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \mid r, s \in S \}.$$

Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R^{(\delta)}$ and

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} R^{(\delta)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0,$$

we see that $R^{(\delta)}$ is not semiprime.

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Department of Mathematics, DePaul University, Chicago, Illinois 60614
E-mail address: `jbergen@condor.depaul.edu`

Institute of Mathematics, University of Warsaw, Białystok Division Akademicka 2, 15-267, Białystok, Poland
E-mail address: `piotrgr@cksr.ac.bialystok.pl`