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RIGID SETS AND NONEXPANSIVE MAPPINGS

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Abstract. We introduce a new class of normed spaces (not necessarily finite dimensional), which contains the finite dimensional normed spaces with polyhedral norm.

We study the properties of rigid sets of the spaces of this class and we apply the results to limit sets of the sequences of iterates of nonexpansive maps.

1. Introduction

Let us consider a compact subset C of a normed space X and a nonexpansive map $f : C \to C$. Let us denote by $\omega(x)$ the set of limit points of sequence $\{f^n(x)\}$, $x \in C$.

In 1987 Akcoglu and Krengel [1], by using some properties of rigid sets, proved that: *if* $X = (\mathbb{R}^n, \| \ \|_1)$, $n \geq 1$, *then* $\omega(x)$ *is finite*, $\forall x \in C$. By a different procedure, Weller [13], extended the above result to any finite dimensional normed space with polyhedral norm. Successively, other authors [5]–[12] also gave an estimate of cardinality of $\omega(x)$ in all these spaces.

In this paper we introduce a suitable class of normed spaces which are not necessarily finite dimensional. We study rigid sets in these spaces and prove that basic properties of rigid sets in $(\mathbb{R}^n, \|\cdot\|_1)$ still hold in the spaces of the above class. This allows us to obtain some results concerning $\omega(x)$. In particular, Theorem 4.1 gives a new simpler proof of Weller's result, quoted before.

2. The S-spaces

We say that a real normed space $(X, \| \ \|)$ is an S-space if the following condition holds:

(P) For each $a \in X - \{0\}$ there exists $\delta = \delta(a) > 0$ such that if $x, y \in X$ and

(i) $\|x\| < \delta, \|y\| < \delta$,
(ii) $\|a - x\| \leq \|a\|, \|a - y\| \leq \|a\|$,
(iii) $\|a - (x + y)\| = \|a\|$,

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then

$$\|a-x\| = \|a\|, \qquad \|a-y\| = \|a\|. \tag{2.1}$$

Proposition 2.1. *If there exists a sequence $\{\varphi_n\}$ of continuous linear functionals on $(X, \| \ \|)$ weakly convergent to zero and*

$$\|x\| = \sup_n |\varphi_n(x)| \quad \forall x \in X,$$

then $(X, \| \ \|)$ is an S-space.

Proof. Let a be a point of $X - \{0\}$. Let us set

$$\delta = \delta(a) = \frac{1}{4}\min\{\|a\|, \min\{\|a\| - |\varphi_n(a)| : |\varphi_n(a)| < \|a\|\}\},$$

and let us assume that x and y belong to X and satisfy (i), (ii), (iii).

If $|\varphi_n(a)| < \|a\|$ we have

$$\begin{aligned} |\varphi_n(a) - \varphi_n(x+y)| &\le |\varphi_n(a)| + 2\delta = |\varphi_n(a)| + 2 \cdot \frac{1}{4}(\|a\| - |\varphi_n(a)|) \\ &= \frac{1}{2}\|a\| + \frac{1}{2}|\varphi_n(a)| < \|a\|. \end{aligned}$$

Then, by (iii), there exists m such that

$$|\varphi_m(a)| = \|a\| \quad \text{and} \quad |\varphi_m(a) - \varphi_m(x+y)| = \|a\|.$$

Thus

$$\varphi_m(x) = -\varphi_m(y),$$

and, from (ii), we obtain

$$\varphi_m(x) = \varphi_m(y) = 0,$$

so (2.1) holds.

Remarks. 1. Hypothesis "$\{\varphi_n\}$ *weakly convergent to zero*" cannot be weakened. Indeed, let us consider the space c of the convergent sequences of real numbers with sup-norm. Let $a = \{(1 - \frac{1}{n})\}_{n \in N}, \delta > 0, n_0$ an integer greater than $(2\delta)^{-1}$, $x = y = \{x_n\}_{n \in N}$ where

$$x_n = \begin{cases} \delta(1 - \frac{1}{n}), & n \ne n_0, \\ -\frac{1}{2n_0}, & n = n_0. \end{cases}$$

It is easy to see that conditions (i), (ii), (iii) hold, but:

$$\|a - x\| < 1 = \|a\|.$$

2. The space $c_0 \subset c$ of the sequences convergent to zero is an S-space.

3. A real finite dimensional normed space endowed with a polyhedral norm is an S-space.

Proposition 2.2. *Let $(X, \| \ \|)$ be an S-space. If $a, x \in X, a \ne 0, \|x\| < \delta(a)$ and $\|a - x\| = \|a\|$, then*

$$\|a - \lambda x\| = \|a\| \quad \forall \lambda \in [0, 1]. \tag{2.2}$$

Indeed:

$$\|a-\lambda x\| = \|\lambda(a-x)+(1-\lambda)a\| \le \|a\|,$$
$$\|a-(1-\lambda)x\| = \|(1-\lambda)(a-x)+\lambda a\| \le \|a\|,$$
$$\|a-[\lambda x+(1-\lambda)x]\| = \|a-x\| = \|a\|.$$

Moreover $\|\lambda x\| \le \|x\| < \delta(a)$ and $\|(1-\lambda)x\| \le \|x\| < \delta(a)$; then, from (P), (2.2) holds.

Corollary 2.1. *An S-space is not necessarily strictly convex.*

Indeed, if $\|a\| = \|b\| = 1$ and $\|a-b\| < \delta(a)$, (2.2) gives $\|a-\frac{1}{2}(a-b)\| = \|\frac{a+b}{2}\| = 1$.

Remark. In an S-space the function $a \to \delta(a)$ can be discontinuous.

Indeed, in $(\mathbb{R}^2, \| \ \|_\infty)$ if $a = (1-\varepsilon, 1)$, $0 < \varepsilon < 1$, we have $\delta(a) \le \varepsilon \to 0$ if $\varepsilon \to 0$.

Theorem 2.1. *If $(X, \| \ \|)$ is a finite dimensional real space, then the following statements are equivalent*:

(1) *For each $a \in X$ there exists $\delta(a) > 0$ such that*:

$$\|a\| = \|b\| \text{ and } \|b-a\| < \delta(a) \Rightarrow \|\lambda b+(1-\lambda)a\| = \|a\|, \ \forall \lambda \in]0,1[.$$

(2) *$\| \ \|$ is a polyhedral norm.*

(3) *$(X, \| \ \|)$ is an S-space.*

In order to prove the above theorem we need:

Lemma 2.1. *Let K be a nonempty compact convex subset of Euclidean space $\mathbb{R}^n$, $n \ge 2$, and let Γ be its boundary. If, for each $a \in \Gamma$, there exists $\delta(a) > 0$ such that*

$$b \in \Gamma \text{ and } \|b-a\| < \delta(a) \Rightarrow \lambda b+(1-\lambda)a \in \Gamma, \ \forall \lambda \in]0,1[, \tag{2.3}$$

then K is a polytope (of dimension $\le n$).

Indeed, if $n = 2$ and K is not a singleton, being Γ compact, there exists a finite number $m \ge 1$ of points of Γ, namely $a_1, a_2, \dots, a_m$, such that the open balls

$$S_i = \{x \in \mathbb{R}^n : \|x-a_i\| < \delta(a_i)\}, \qquad i = 1, \dots, m,$$

are a finite covering of Γ. So, by (2.3), $\Gamma \cap S_i$ is constituted by at most two segments. Then Γ is either a segment or the boundary of a convex polygon (endowed with at most m edges).

If $n > 2$, we go on by induction.

Let $a_1, a_2, \dots, a_m$, $S_1, S_2, \dots, S_m$ be as above.

By (2.3), for every $i = 1, \dots, m$, $\Gamma \cap S_i = \gamma_i \cap S_i$, where γ_i is the boundary of a (convex) cone Γ_i obtained by projecting from a_i the intersection of K with a suitable hyperplane of $\mathbb{R}^n$.

Every nonempty intersection K' of K with a hyperplane of $\mathbb{R}^n$ lies in $\mathbb{R}^{n-1}$ and satisfies the hypotheses of the lemma; so K' is a polytope of dimension $\le n-1$.

Then Γ_i is a polyhedral cone (which in some case can degenerate) and K is a polytope of dimension $\le n$ (with at most m vertices). q.e.d.

Remark. In the case $n > 2$ Lemma 2.1 can be proved also by using a result of Klee ([4], Theorem 4.7).

Proof of Theorem 2.1. (1) $\Rightarrow$ (2) The unit ball U of X is isomorphic to a subset K of an euclidean space $\mathbb{R}^n$ which satisfies hypotheses of Lemma 2.1. Then $\| \ \|$ is a polyhedral norm.

(2) $\Rightarrow$ (3) By Proposition 2.1.

(3) $\Rightarrow$ (1) By Proposition 2.2.

Corollary 2.2. *An S-space is not smooth.*

Indeed, by Theorem 2.1, the intersection of its unit ball with a two-dimensional linear subspace of X is not smooth. Then X is not smooth ([2], (2), p. 112).

3. Rigid sets in an S-space

Let us recall that a compact set in a metric space (M, d) is called "*rigid*" if it is the closure of a sequence $\{x_n\}$ in M such that

$$d(x_{n+k}, x_n) = d(x_{k+1}, x_1), \quad \forall n \geq 1, \ \forall k \geq 1.$$

Rigid sets were studied in the framework of metric spaces by Akcoglu and Krengel [1]; in particular they proved that: *rigid sets in* $(\mathbb{R}^n, \| \ \|_1)$ *are finite.*

In this section, we extend this result to rigid sets in any finite dimensional S-space.

Let us start by proving:

Lemma 3.1. *Let A be a rigid set in an S-space, $d = \operatorname{diam} A$ and $x, y \in A$ such that*

$$\|x - y\| = d.$$

Then there exists $\varepsilon = \varepsilon(x - y)$ such that

$$z \in A \text{ and } \|y - z\| < \varepsilon \ \Rightarrow \ \|x - z\| = d. \tag{3.1}$$

Proof. If $d = 0$, (3.1) is trivial. If $d > 0$, there is an isometry $T : A \to A$ such that $\{T^n(y)\}$ is dense in A (see [1], Lemma 3.2); so, it suffices to prove that (3.1) holds when $z = T^r(y)$, for some $r > 0$.

Let us set $y - x = a$ and $\varepsilon = \delta(a)$, where $\delta(a)$ is given by (P).

If $\|T^r(y) - y\| < \varepsilon$, from Lemma 3.3 of [1], we also have $\|T^r(x) - x\| < \varepsilon$. Moreover,

$$\begin{aligned}
&\|a - (T^r(x) - x)\| = \|y - x - T^r(x) + x\| = \|y - T^r(x)\| \leq \|a\|,\\
&\|a - (y - T^r(y))\| = \|y - x - y + T^r(y)\| = \|T^r(y) - x\| \leq \|a\|,\\
&\|a - [(T^r(x) - x) + (y - T^r(y))]\| = \|y - x - T^r(x) + x - y + T^r(y)\|\\
&\qquad = \|T^r(y) - T^r(x)\| = \|y - x\| = \|a\|.
\end{aligned}$$

Hence, from (P), we have

$$\|T^r(y) - x\| = \|a\| = d.$$

Thus the assertion has been proved.

Remark. Lemma 3.1 generalizes Lemma 3.8 of [1] to any S-space (both finite and infinite dimensional).

Corollary 3.1. *With the hypotheses of Lemma 3.1 the set*

$$\{z \in A : \|z - x\| = d\}$$

is open on A.

Theorem 3.1. *A rigid set in a finite dimensional S-space is finite.*

Proof. Let A_0 be a rigid set. Since A_0 is compact, in order to prove this theorem, it suffices to show that A_0 is totally disconnected.

Let a be a point of A_0. By Lemma 3.4 of [1] there exists $a_0 \in A_0$ such that $\|a_0 - a\| = \operatorname{diam} A_0$.

Let us set

$$B_1 = \{x \in A_0 : \|x - a_0\| = \operatorname{diam} A_0\}.$$

B_1 is a closed set, B_1 is open on A_0 by Corollary 3.1 and $a \in B_1$.

So, by Corollary 3.7 of [1], there exists a rigid set A_1 open on A_0, such that $a \in A_1 \subset B_1$.

Then, iterating the above procedure, we obtain a sequence $\{a_n\}$ of points of A_0 and a sequence $\{A_n\}$ of rigid sets, open on A_0, such that for each $n \geq 0$

$$a_n \in A_n, \quad \|a_n - a\| = \operatorname{diam} A_n,$$
$$a \in A_{n+1} \subset A_n, \quad \|x - a_n\| = \operatorname{diam} A_n \quad \forall x \in A_{n+1}.$$

Hence the sequence $\{x_n\}$, where $x_n = a - a_n$, is such that

$$\|x_m - x_n\| = \|x_n\| \quad \forall n \geq 0 \ \forall m > n.$$

Then, by Theorem 2 of [3], there exists h such that $\|x_h\| = 0$; consequently $A_h = \{a\}$. So, since A_h is open on A_0, a is an isolated point of A_0.

Thus the assertion has been proved.

Remark. Theorem 3.1 is not true any longer if the S-space is infinite dimensional. It is possible to give different counterexamples; we report a simple one given by the referee.

Given the S-space c_0, as an infinite rigid set of c_0 we can consider the ω-limit set of the sequence $\{f^n(\{\frac{1}{k}\})\}, \{\frac{1}{k}\} \in c_0$ and $f : c_0 \to c_0$ permutes the first two, the next three,..., the next $p_1, \ldots$ components, where p_i is the i-th prime number.

4. Omega limit sets of nonexpansive mappings in an S-space

As a consequence of Theorem 3.1 we have

Theorem 4.1. *Let C be a compact subset of a finite dimensional S-space and $f : C \to C$ be a nonexpansive map. Then, for each $x_0 \in C$, $\omega(x_0)$ is finite.*

Indeed, $\forall x_0 \in C$, $\omega(x_0)$ is a rigid set because the restriction of f to $\omega(x_0)$ is an isometry from $\omega(x_0)$ onto $\omega(x_0)$ such that, $\forall x \in \omega(x_0)$, the sequence $\{f^n(x)\}$ is dense in $\omega(x_0)$ (see [7], p. 523). So, according to Lemma 3.2 of [1], $\omega(x_0)$ is a rigid set.

Corollary 4.1. *Let C be a subset of an S-space and $f : C \to C$ be a nonexpansive map. If $x_0 \in C$ has bounded orbit and $\omega(x_0)$ lies in a finite dimensional linear subspace of X, then $\omega(x_0)$ is finite.*

Indeed, $\omega(x_0)$ is a closed bounded subset of a finite dimensional S-space and $f(\omega(x_0)) = \omega(x_0)$. Then $\omega(x_0)$ is compact and $\omega(y) = \omega(x_0)$ $\forall y \in \omega(x_0)$. Hence, we can apply Theorem 4.1 to any point y of the compact $\omega(x_0)$ and we obtain the result.

Corollary 4.1 is a local version of Theorem 4.1 which holds also in infinite dimensional S-spaces. In general, in these spaces we have only the following result:

Theorem 4.2. *Let C be a compact subset of an S-space and $f : C \to C$ be a nonexpansive map. Then, for each $x_0 \in C$, $\omega(x_0)$ is either a singleton or not connected.*

Proof. Let x_0 be a point of C such that $\omega(x_0)$ is not a singleton. Since $\omega(x_0)$ is a rigid set, $\forall a \in \omega(x_0)$, the closed set

$$H = \{x \in \omega(x_0) : \|a - x\| = \operatorname{diam} \omega(x_0)\}$$

is open on $\omega(x_0)$ by Corollary 3.1.

Then the set $K = \omega(x_0) - H$ is closed and K is nonempty because $a \in K$. Then $\omega(x_0)$ is not connected.

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